Two-level type theory in Lean

Semi-simplicial types in homotopy type theory

Adrien Mathieu

Thursday 12th, September

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bijection	equivalence

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The solution is simple: *make the equality equivalent to the equivalences*!

$$(A \simeq B) \simeq (A = B)$$

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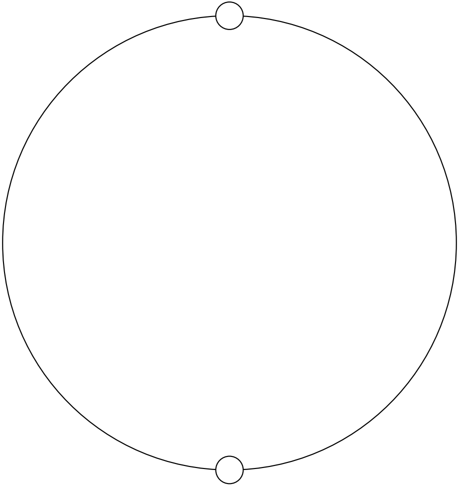

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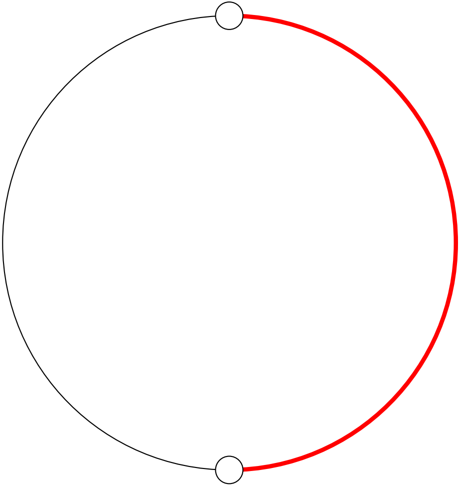

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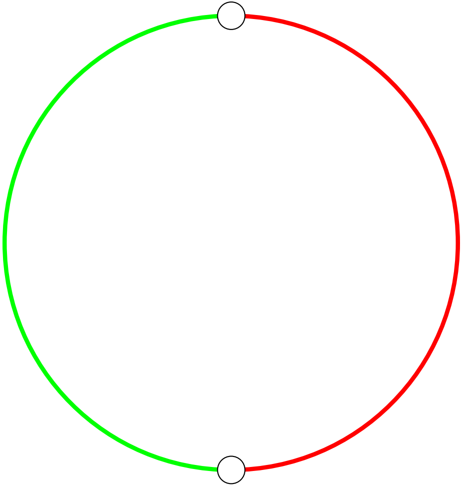

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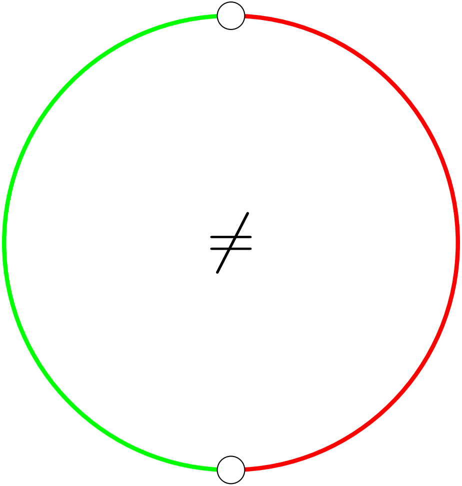

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Let's start with semi-simplicial types.

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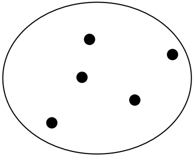

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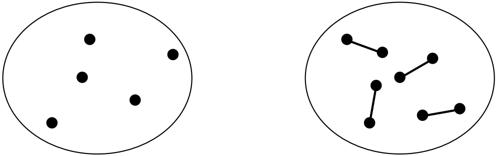

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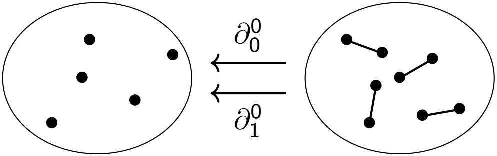

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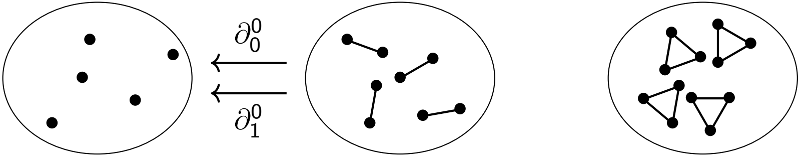

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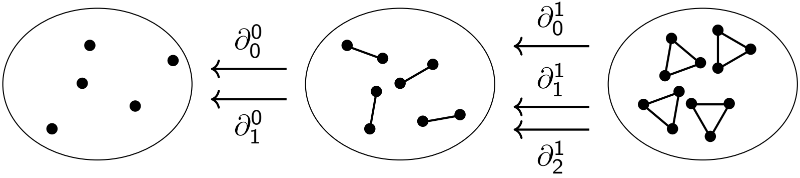

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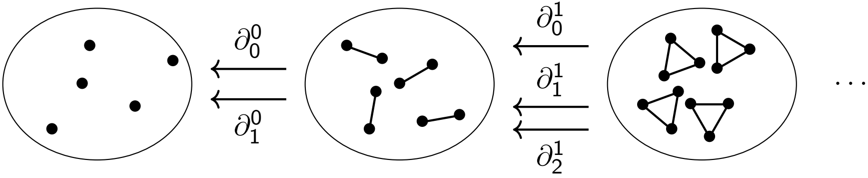

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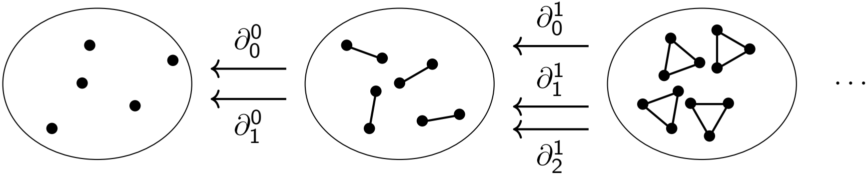

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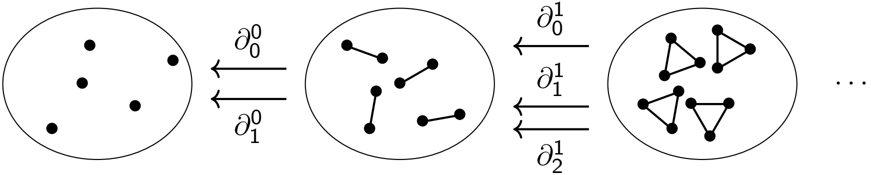

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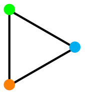

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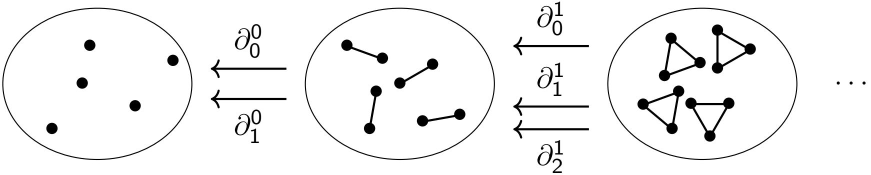

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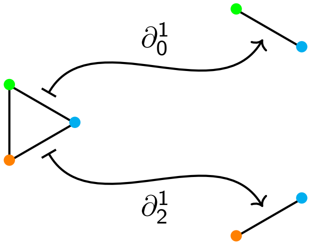

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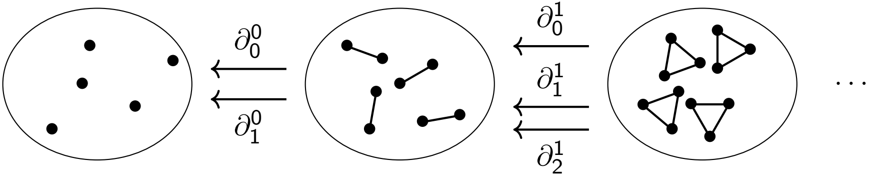

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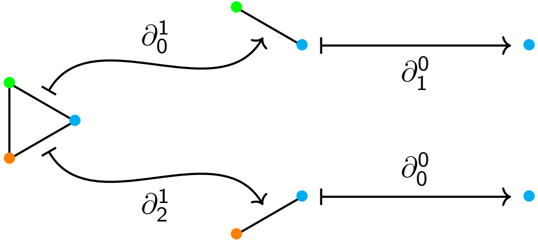

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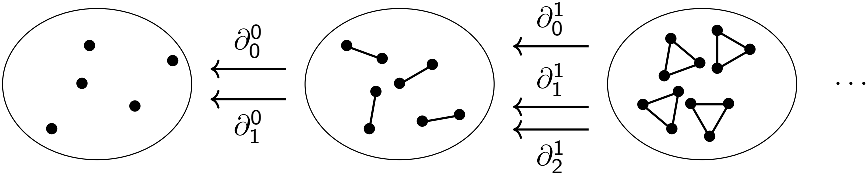

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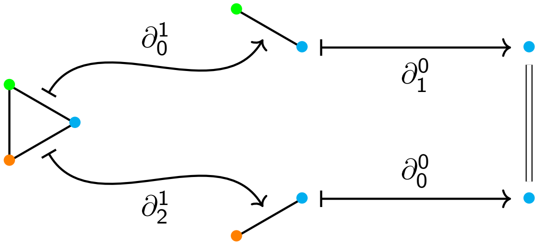


Formal specification

A semi-simplicial type is the data of

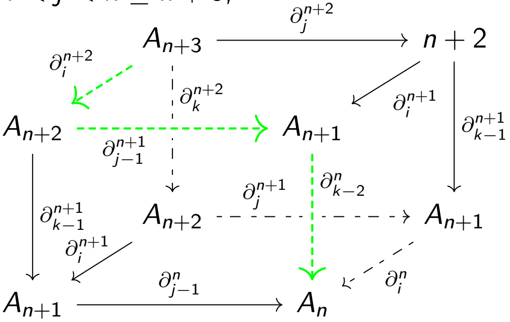
$$(A_n)_{n:\mathbb{N}} : \mathbb{N} \to \mathsf{Type}$$
$$\partial_i^n : A_{n+1} \to A_n \qquad \text{for } n : \mathbb{N} \text{ and } i \leq n+1$$

such that, for $n : \mathbb{N}$ and $i < j \leq n+1$,

$$\begin{array}{ccc} A_{n+2} & \xrightarrow{\partial_j^{n+1}} & A_{n+1} \\ \Big\downarrow{\scriptstyle \partial_i^{n+1}} & & \Big\downarrow{\scriptstyle \partial_i^{n}} \\ A_{n+1} & \xrightarrow{\partial_{j-1}^{n}} & A_n \end{array}$$

Higher problems

Given $n : \mathbb{N}$, and $i < j < k \leq n + 3$,



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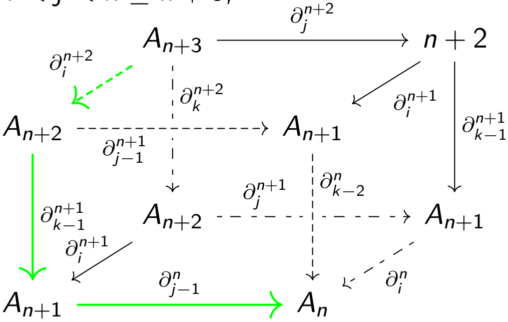

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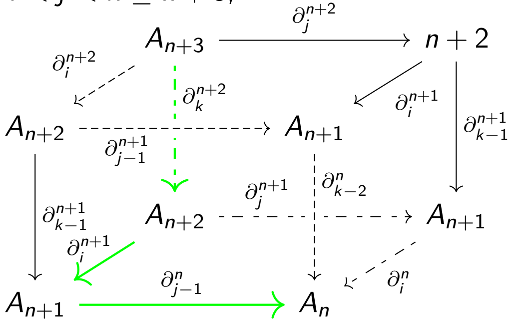

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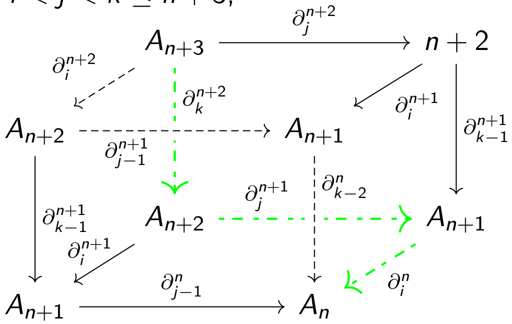

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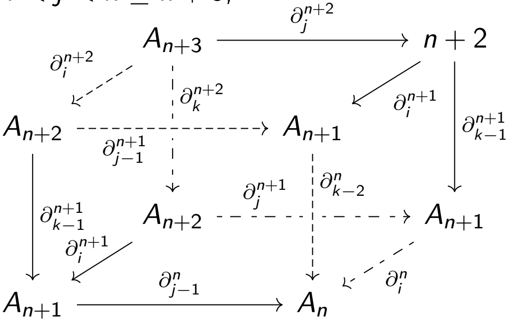

$$p : \partial_{k-2}^{n} \circ \partial_{j-1}^{n+1} \circ \partial_i^{n+2} = \partial_i^{n} \circ \partial_j^{n+1} \circ \partial_k^{n+2}$$

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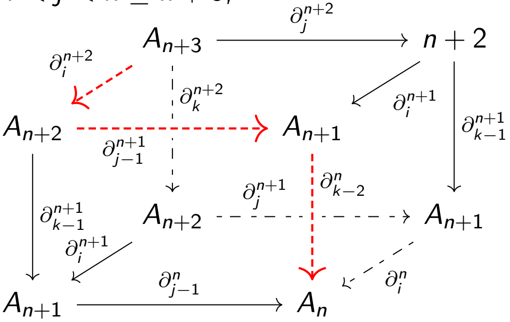

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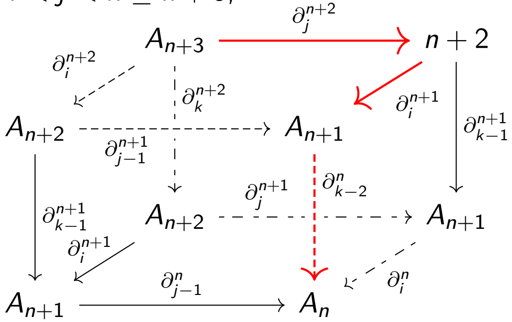

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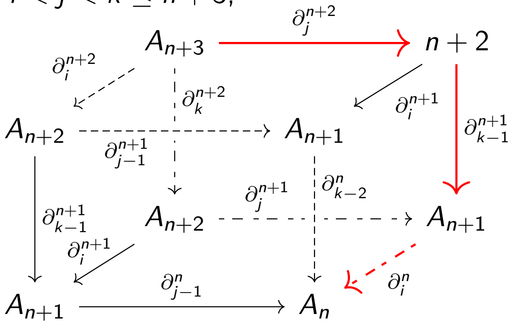

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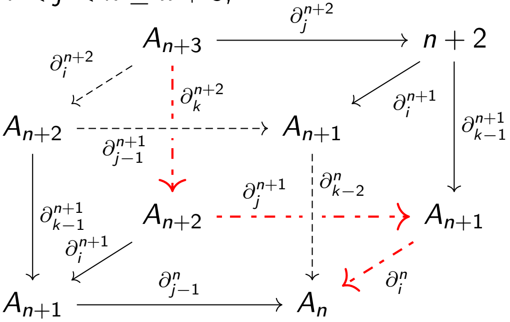

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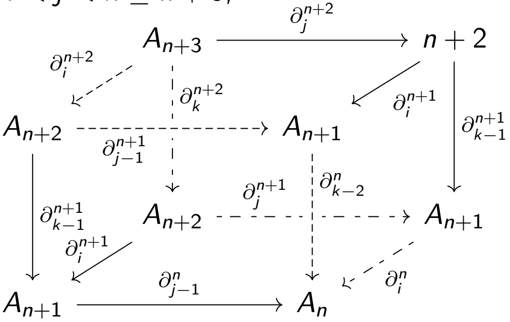

$$p : \partial^n_{k-2} \circ \partial^{n+1}_{j-1} \circ \partial^{n+2}_i = \partial^n_i \circ \partial^{n+1}_j \circ \partial^{n+2}_k$$

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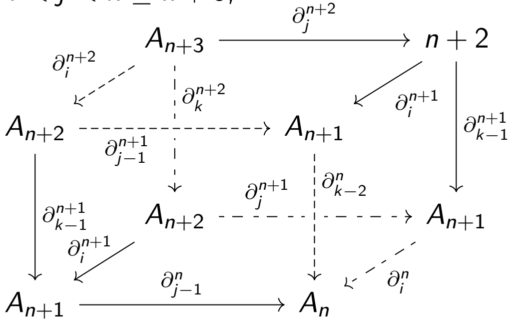

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$$p \stackrel{?}{=} q$$

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It is impossible to define semi-simplicial types.

But this is self-contradictory!

The meta trick

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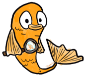
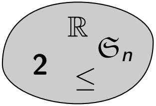


$n + 1 = 1 + n$



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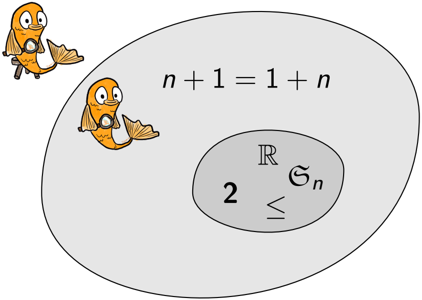

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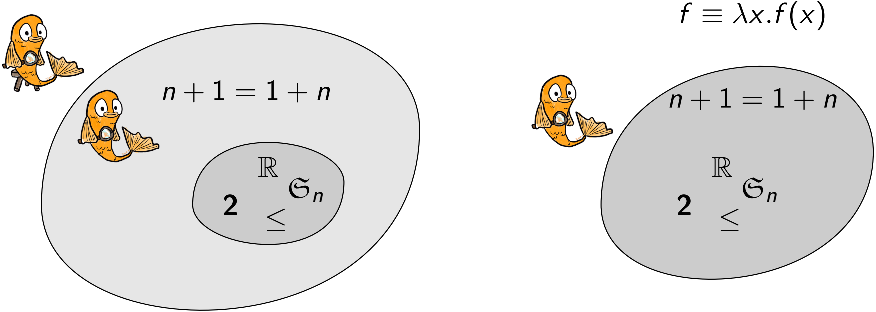

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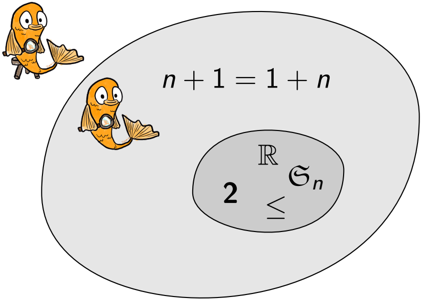

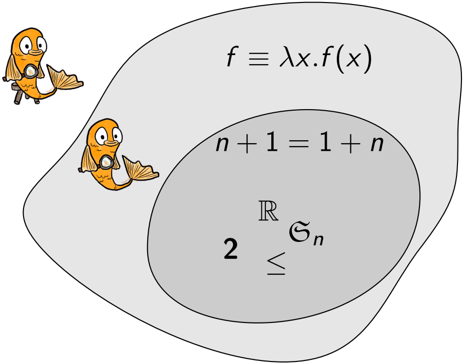

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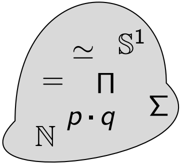

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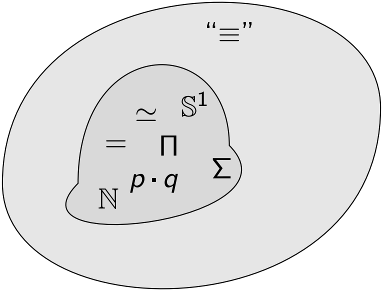

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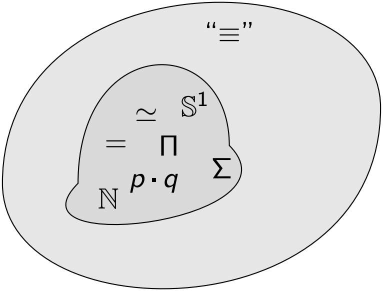
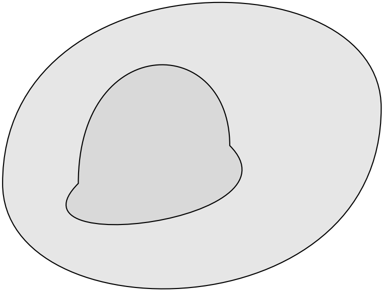


Homotopy type theory can be lifted in the two-level type theory.

Stating the conjecture

We now have a strategy to state the conjecture:



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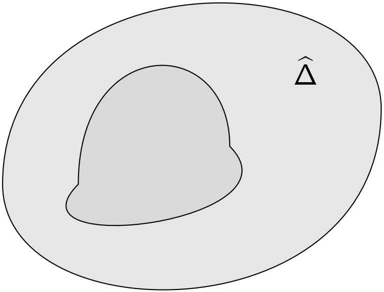

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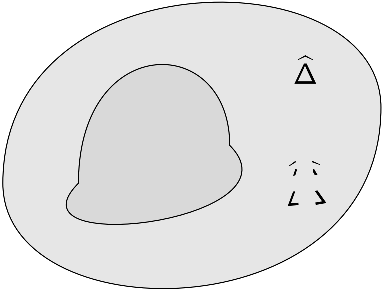

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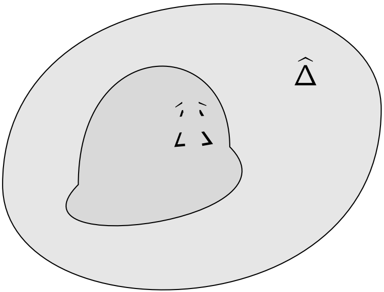

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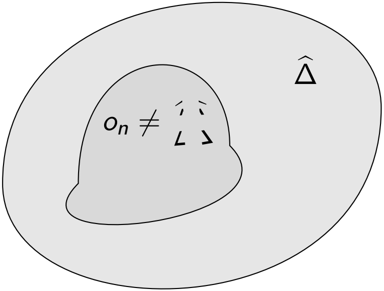
- state "there is no collection of objects in HoTT that is pointwise equal to the collection of fragments"



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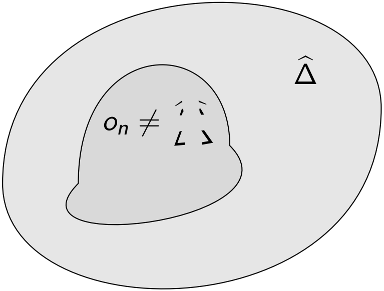

If there was such a think such as semi-simplicial types directly in HoTT, we could also break it

The Reedy way (1/2)

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$$\left[\begin{array}{l} A_0 : \text{Type} \\ A_1 : A_0 \to A_0 \to \text{Type} \end{array}\right.$$

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$$
\begin{aligned}
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&A_1 : A_0 \to A_0 \to \mathsf{Type} \\
&A_2 : (x : A_0) \to (y : A_0) \to (z : A_0) \\
&\qquad\qquad \to A_1(x, y) \to A_1(x, z) \to A_1(y, z) \to \mathsf{Type}
\end{aligned}
$$

The Reedy way (1/2)

We now have to break semi-simplicial types into fragments that fit into HoTT. To do so, we present semi-simplicial types in the Reedy fashion.

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&A_3 : (x, y, z, w : A_0) \to (s_1 : A_1(x, y)) \to (s_2 : A_1(x, z)) \to (s_3 : A_1(x, w))\\
&\qquad\quad \to (s_4 : A_1(y, z)) \to (s_5 : A_1(y, w)) \to (s_6 : A_1(z, w))\\
&\qquad\quad \to A_2(x, y, z, s_1, s_2, s_4) \to A_2(x, y, w, s_1, s_3, s_5)\\
&\qquad\quad \to A_2(x, z, w, s_2, s_3, s_6) \to A_2(y, z, w, s_4, s_5, s_6)\\
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\qquad\quad \to \mathrm{Type} \\
\qquad \vdots
\end{array}
\right.
$$

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These objects are called *very dependent types*. It is unknown whether they can be formulated directly in HoTT, and even whether they are consistent in it!

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But we can generate the signature of each level using the meta language.
The fragments of the semi-simplicial types are simply the truncation up to A_n, for every $n : \mathbb{N}$.
The bulk of the work is showing that these fragments "fit" in HoTT.