Notes on Representation of 2-groups

Adrien Mathieu

Contents

1. Preliminary notions

1.1. 2-categories

Definition 1.1.1 (Strict 2-category)

A (strict) 2-category is a **Cat**-enriched category, that is, a category $\mathcal{C}$ such that, for each two objects $X, Y \in \mathcal{C}$, $\mathcal{C}(X,Y)$ is a category, such that the composition $\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ is a functor.

♣

Definition 1.1.2 (Strict 2-functor)

A (strict) 2-functor between two 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$ seen as 1-categories, with an additional structure: for each two objects $X, Y \in \mathcal{C}$, $F_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ is a functor and such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) & \xrightarrow{F_{X,Y} \times F_{Y,Z}} & \mathcal{D}(F(X),F(Y)) \times \mathcal{D}(F(Y),F(Z)) \\ \downarrow * & & \downarrow * \\ \mathcal{C}(X,Z) & \xrightarrow{F_{X,Z}} & \mathcal{D}(F(X),F(Z)) \end{array}$$

♣

1.2. Categories of internal categories

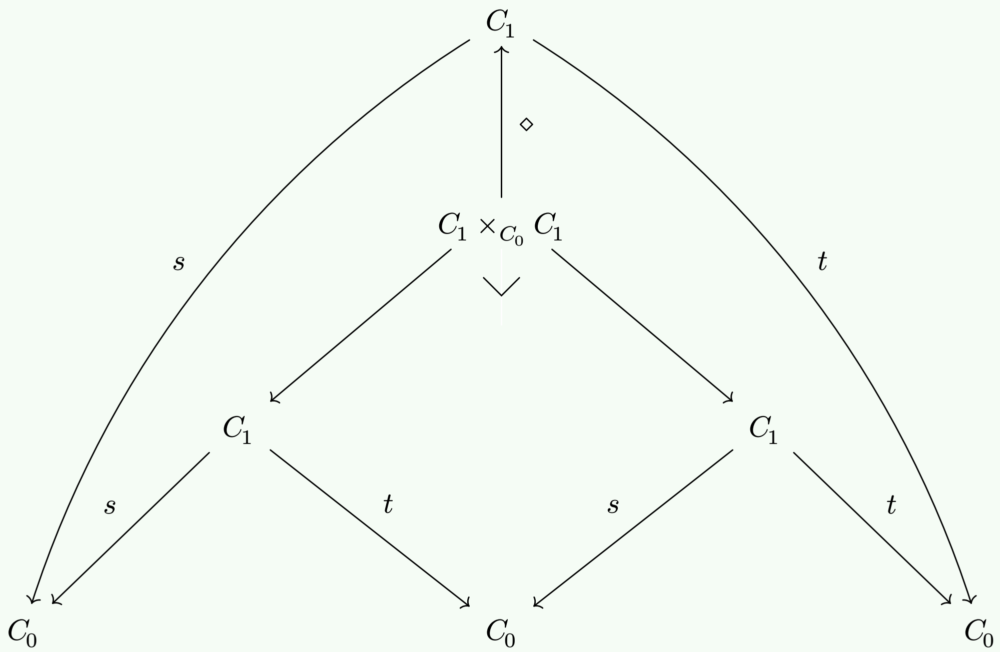
Definition 1.2.1 (Internal category)

Let $\mathcal{C}$ be a category. An *internal category* in $\mathcal{C}$ is the datum of two objects C_0 and C_1 in $\mathcal{C}$, as well as two morphisms $s, t : C_1 \to C_0$, named *source* and *target*.

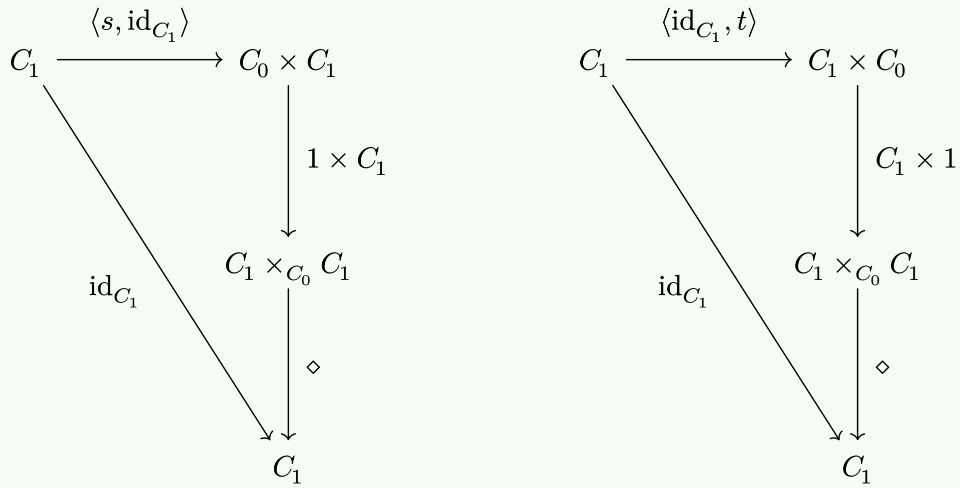
Furthermore, there is an application $1 : C_0 \to C_1$ making the following diagram commute

$$\begin{array}{ccccc} & & C_1 & & \\ & \swarrow^{s} & \uparrow 1 & \searrow^{t} & \\ C_0 & \xleftarrow{\mathrm{id}_{C_0}} & C_0 & \xrightarrow{\mathrm{id}_{C_0}} & C_0 \end{array}$$

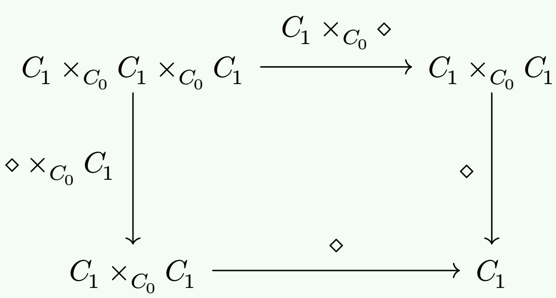
as well as an application $\diamond : C_1 \times_{C_0} C_1 \to C_1$ making the following diagram commute



such that the two following diagrams commute



as well as the following one



♣

Definition 1.2.2 (Internal functor)

Given two internal categories C and D in $\mathcal{C}$, an *internal functor F from C to D* is the datum of a morphism

$$F_0 : C_0 \to D_0$$

as well as a morphism

$$F_1 : C_1 \to D_1$$

making the following diagrams commute

$$\begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ C_0 & \xrightarrow{F_0} & D_0 \end{array} \qquad \begin{array}{ccc} C_0 & \xrightarrow{F_0} & D_0 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ C_1 & \xrightarrow{F_1} & D_1 \end{array} \qquad \begin{array}{ccc} C_1 & \xrightarrow{F_1} & D_1 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ C_0 & \xrightarrow{F_0} & D_0 \end{array}$$

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{F_1 \times F_1} & D_1 \times_{D_0} D_1 \\ {\scriptstyle \diamond}\downarrow & & \downarrow{\scriptstyle \diamond} \\ C_1 & \xrightarrow{F_1} & D_1 \end{array}$$

Fact 1.1

For two internal functors $F : C \to D$ and $G : D \to E$, there exists a functor

$$G \circ F : C \to E$$

such that $\circ$ is associative.

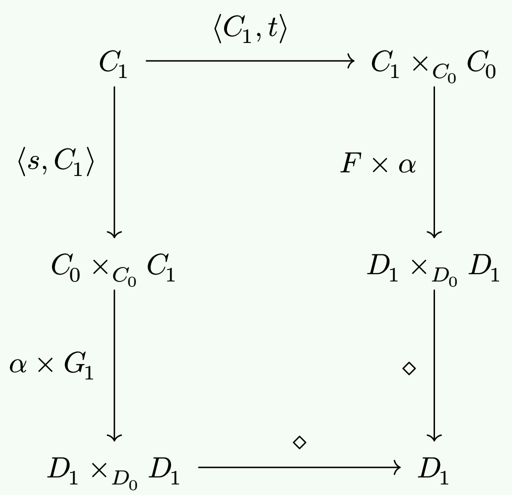
For any internal category C of $\mathcal{C}$, there exists an internal functor $\mathrm{id}_C : C \to C$, that is an identity with respect to $\circ$.

Definition 1.2.3 (Internal natural transformation)

Given two internal categories C and D of $\mathcal{C}$, and two internal functors $F, G : C \to D$, we call and *internal transformation from F to G* a morphism $\alpha : C_0 \to D_1$ such that the following diagrams commute

$$\begin{array}{ccc} C_0 & \xrightarrow{\alpha} & D_1 \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle s} \\ & & D_0 \end{array} \qquad \begin{array}{ccc} C_0 & \xrightarrow{\alpha} & D_1 \\ & {\scriptstyle G}\searrow & \downarrow{\scriptstyle t} \\ & & D_0 \end{array}$$

as well as the naturality diagram



Fact 1.2

The class of internal categories of $\mathcal{C}$, with internal functors as morphisms and internal natural transformations as 2-cells, forms a 2-category, noted $\mathbf{Cat}_{\mathcal{C}}$.

2. Representation of 2-groups

Let $\mathbb{K}$ be a fixed field for the rest of the section.

2.1. Representation of a group

Let G be a group.

Definition 2.1.1 (Representation of G)

A *representation of* G is a $\mathbb{K}$-linear space V, with a linear action $G \circlearrowright V$.

Given two representations V and W of G, a morphism of representations $V \to W$ is a linear map $f : V \to W$ such that, for every $g \in G$, and $v \in V$, we have

$$f(g \cdot v) = g \cdot f(v)$$

Exercise 2.1

Prove that representations of G forms a category $\mathbf{Rep}_G$

Exercise 2.2

Show that

$$\mathbf{Rep}_G \simeq [G, \mathbf{Vect}_{\mathbb{K}}]$$

where $[G, \mathbf{Vect}_{\mathbb{K}}]$ is the category of functors from G seen as a category, and $\mathbf{Vect}_{\mathbb{K}}$ is the category of linear spaces with linear maps as morphisms.

2.2. 2-group

Definition 2.2.1 (2-group)

A 2-group is a 2-category, with exactly one object, where every 1-cell and 2-cell are isomorphisms. ♣

Definition 2.2.2 (2-representation)

Let $\mathcal{G}$ be a 2-group. A *2-representation of* $\mathcal{G}$ is a 2-functor

$$\mathcal{G} \longrightarrow \mathbf{Cat}_{\mathbf{Vect}_{\mathbb{K}}}$$

♣

Theorem 2.2.3

A 2-group $\mathcal{G}$ is exactly the data of a group G, a group H with a morphism $t : H \to G$, as well as an action $G \circlearrowright H$ such that, for all $g \in G$ and $h \in G$,

$$t(g \cdot h) = gt(h)g^{-1}$$

and for $h, h' \in H$, we have

$$(t(h) \cdot h')h = hh'$$

with $\mathcal{G}_1 = G$ and $\mathcal{G}_2 = H \rtimes G$. ♡

Proof. Suppose we have a group H with a morphism $t : H \to G$, and an action $G \circlearrowright H$. Let us consider the 2-group $\mathcal{G}$ upon G, with 2-cells defined by $\mathcal{G}(g, g') = \{h \in H \mid t(h)g = g'\}$. The identity at $g \in G$ is given by $(1_H, g)$. Let $(h, g) : g \Rightarrow g'$ and $(h', g') : g' \Rightarrow g''$. The vertical composition is given by

$$h' \circ h = h'h : g \Rightarrow g''$$

Indeed,

$$\begin{aligned} t(h'h)g &= t(h')t(h)g \\ &= t(h')g' \\ &= g'' \end{aligned}$$

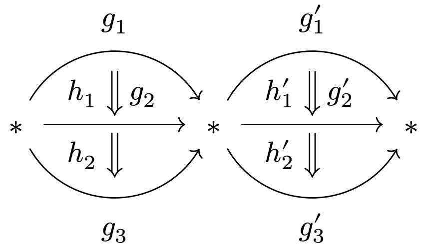
Let $h : g_1 \Rightarrow g_2$ and $h' : g_1' \Rightarrow g_2'$. Their horizontal composition is given by

$$h * h' = h(g_1 \cdot h')$$

Indeed,

$$\begin{aligned} t(h(g_1 \cdot h'))g_1 g_1' &= t(h)t(g_1 \cdot h')g_1 g_1' \\ &= t(h)g_1 t(h')g_1^{-1} g_1 g_1' \\ &= (t(h)g_1)(t(h')g_1') \\ &= g_2 g_2' \end{aligned}$$

Let us now check functoriality, that is, in the following situation

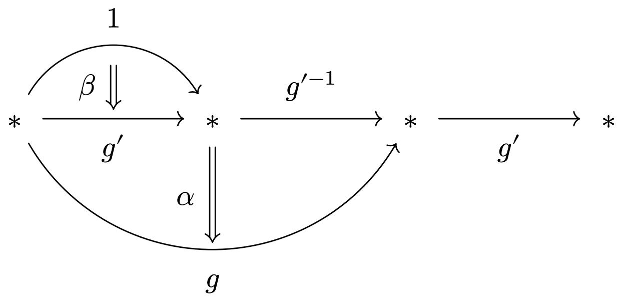


We have

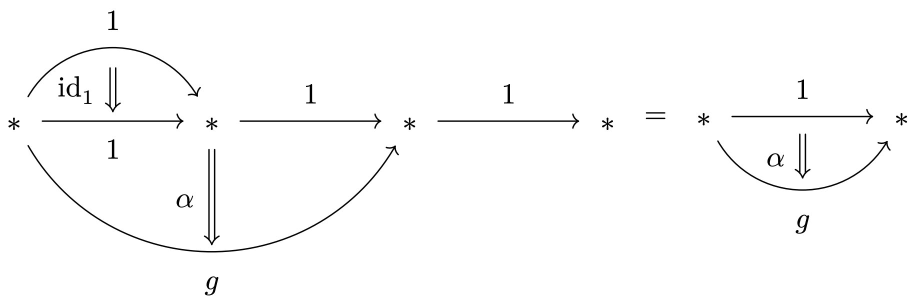
$$\begin{aligned}(h_2 * h'_2) \circ (h_1 * h'_1) &= (h_2 * h'_2)(h_1 * h'_1) \\ &= h_2(g_2 \cdot h'_2)h_1(g_1 \cdot h'_1) \\ &= h_2(t(h_1)g_1 \cdot h'_2)h_1(g_1 \cdot h'_1) \\ &= h_2(t(h_1) \cdot g_1 \cdot h'_2)h_1(g_1 \cdot h'_1) \\ &= h_2h_1(g_1 \cdot h'_2)(g_1 \cdot h'_1) \\ &= h_2h_1(g_1 \cdot (h'_2h'_1)) \\ &= (h_2h_1) * (h'_2h'_1) \\ &= (h_2 \circ h_1) * (h'_2 \circ h'_1)\end{aligned}$$

Conversely, suppose we have a 2-group $\mathcal{G}$. Let us define G be the group of 1-cells of $\mathcal{G}$. Let H be the group of 2-cells of the form $\mathcal{G}(1, g)$ for some g. Let us show that this is a group, for the multiplication given by, for $\alpha : 1 \Rightarrow g$ and $\beta : 1 \Rightarrow g'$,

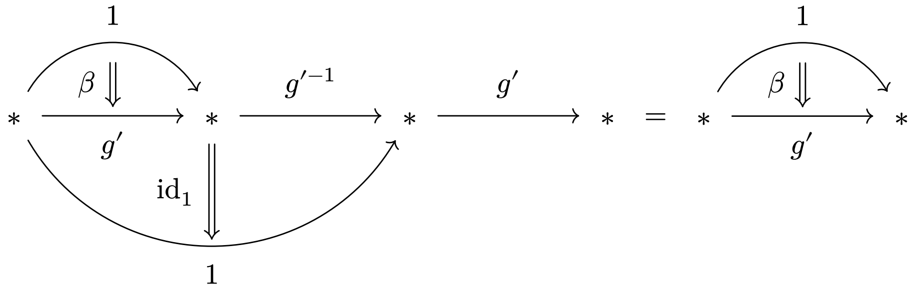
$$\alpha \cdot \beta \coloneqq \left(\alpha \circ \left(\beta * g'^{-1}\right)\right) * g' : 1 \Rightarrow gg'$$



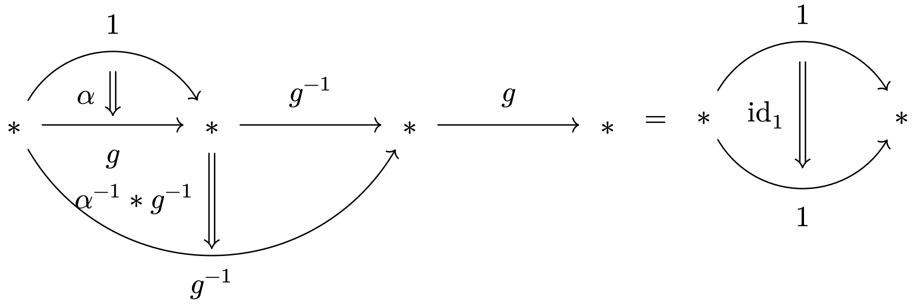
It has a neutral $\mathrm{id}_1 : 1 \Rightarrow 1$, indeed, for $\alpha : 1 \Rightarrow g$, we have



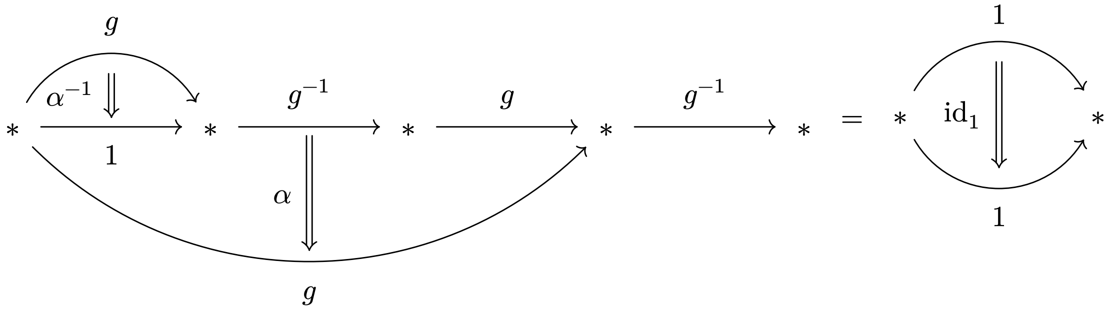
and similarly,



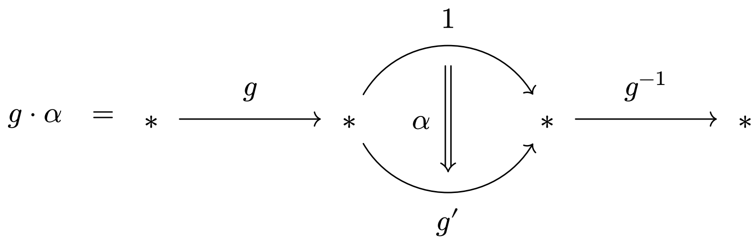
and the inverse of $\alpha : 1 \Rightarrow g$, is $\alpha^{-1} * g^{-1}$. Indeed,



and



Let's now exhibit an action of $G \circlearrowright H$. For $g \in G$ and $\alpha : 1 \Rightarrow g'$,



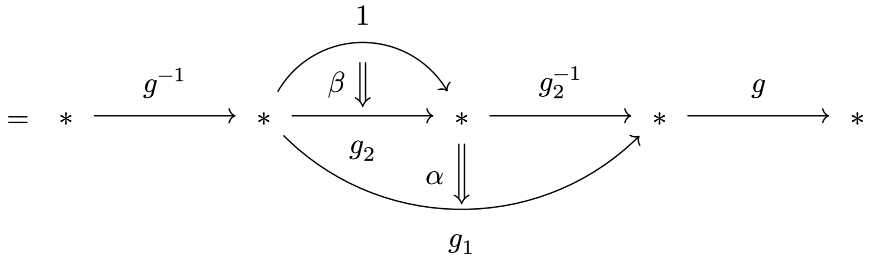
$$= g * \alpha * g^{-1}$$

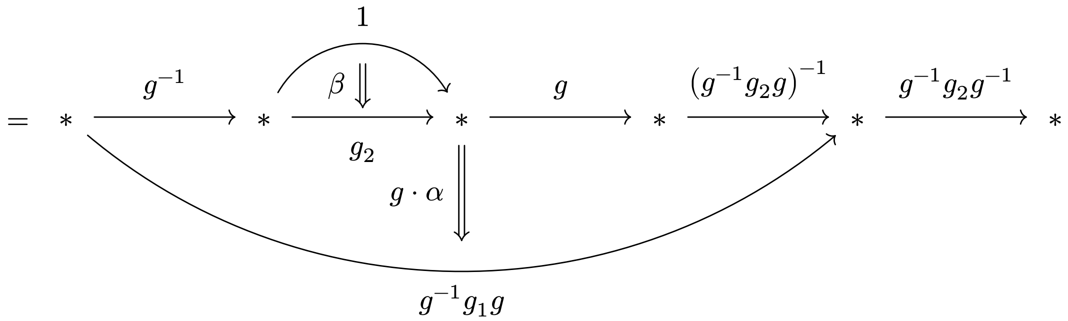
We have

$$\begin{aligned}(g_1 g_2) \cdot \alpha &= (g_1 g_2) * \alpha * (g_1 g_2)^{-1} \\ &= g_1 * g_2 * \alpha * g_2^{-1} * g_1^{-1} \\ &= g_1 \cdot (g_2 \cdot \alpha)\end{aligned}$$

hence it is an action. Let's now show that it is a group morphism: let $g \in G$, $\alpha : 1 \Rightarrow g_1$ and $\beta : 1 \Rightarrow g_2$. We have

$$g \cdot (\alpha \cdot \beta) = g^{-1} * g_2 * (\alpha \circ (\beta * g_2^{-1})) * g$$





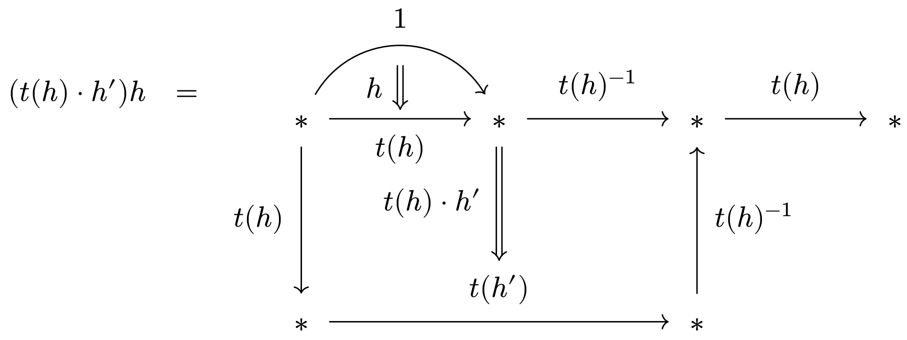
$$= (g \cdot \alpha) \cdot (g \cdot \beta)$$

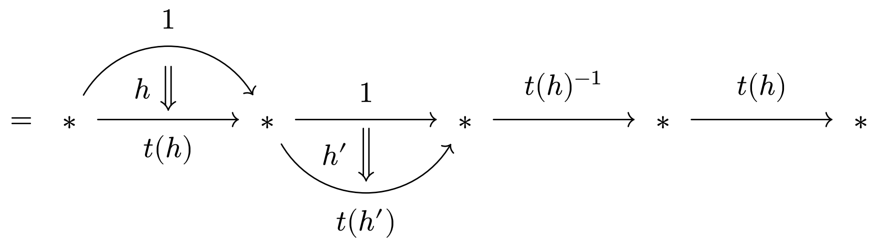
Finally, there is a morphism

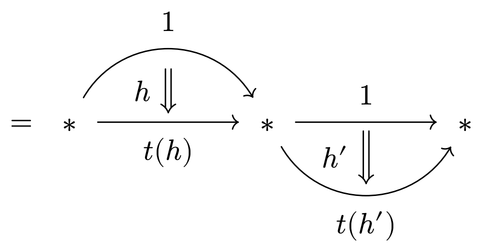
$$t : \quad H \longrightarrow G$$
$$(\alpha : 1 \Rightarrow g) \longmapsto g$$

Let's check that, for $h, h' \in H$, we have

$$(t(h) \cdot h')h = hh'$$







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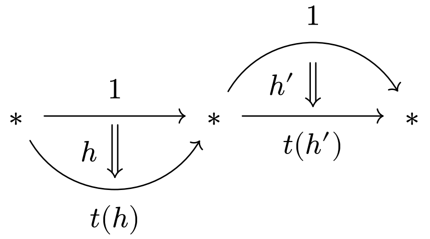

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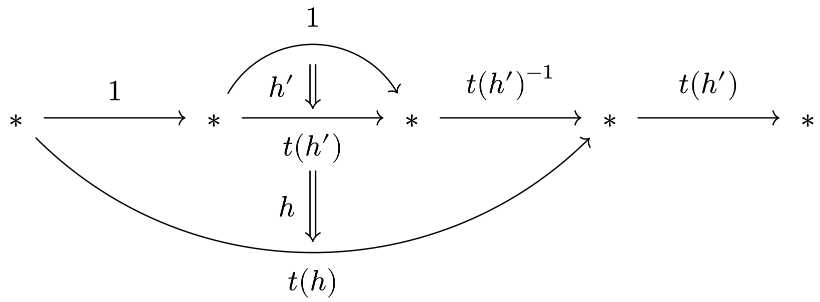

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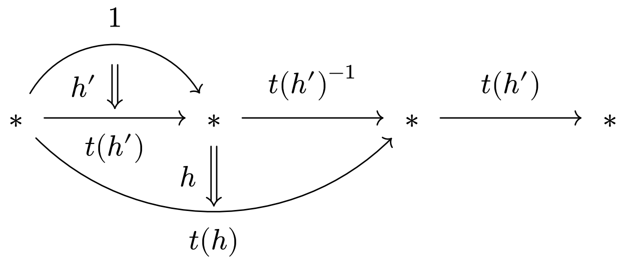

$= hh'$

□

2.3. 2-representation of a 2-group

2.3.1. From the groupoid perspective

Let $\mathcal{G}$ be a fixed 2-group, and $F : \mathcal{G} \to \mathbf{Cat}_{\mathbf{Vect}_{\mathbb{K}}}$ be a 2-representation. This is the data of an internal category $V \coloneqq V_1 \rightrightarrows V_0$ in $\mathbf{Vect}_{\mathbb{K}}$, such that, for every $g \in \mathcal{G}_1$, there is a functor

$$g \cdot - : V \to V$$

that is, linear maps $g \cdot_0 - : V_0 \to V_0$ and $g \cdot_1 - : V_1 \to V_1$ such that the following equations hold

$$\begin{aligned}
g \cdot_1 1_v &= 1_v && \text{for every } v \in V_0\\
s(g \cdot_1 f) &= g \cdot_0 s(f) && \text{for every } f \in V_1\\
t(g \cdot_1 f) &= g \cdot_0 t(f) && \text{for every } f \in V_1\\
g \cdot_1 (f' \diamond f) &= (g \cdot_1 f') \diamond (g \cdot_1 f) && \text{for every } f, f' \in V_1 \text{ such that } t(f') = s(f)
\end{aligned}$$

The 1-functoriality of F expresses exactly that $g \circlearrowright V_0$ and $g \circlearrowright V_1$, that is, the following equations hold:

$$\begin{aligned}
e \cdot_0 v &= v && \text{for } v \in V_0\\
e \cdot_1 f &= f && \text{for } f \in V_1\\
(gg') \cdot_0 v &= g \cdot_0 (g' \cdot_0 v) && \text{for } v \in V_0\\
(gg') \cdot_1 f &= g \cdot_1 (g' \cdot_1 f) && \text{for } f \in V_1
\end{aligned}$$

Let's now look at the 2-functoriality part of F. For $g, g' \in G$ elements of G, and $\alpha : g \Rightarrow g'$ a 2-cell. We have that

$$\alpha \cdot - : g \cdot_1 - \Rightarrow g' \cdot_1 -$$

is an internal natural transformation, that is, a linear map $\alpha \cdot - : V_0 \to V_1$ such that

$$\begin{aligned} s(\alpha \cdot v) &= g \cdot_0 v && \text{for } v \in V_0 \\ t(\alpha \cdot v) &= g' \cdot_0 v && \text{for } v \in V_0 \\ (\alpha \cdot t(f)) \diamond (g \cdot_1 f) &= (g' \cdot_1 f) \diamond (\alpha \cdot s(f)) && \text{for every } f \in V_1 \end{aligned}$$

The 2-functoriality implies that, for every $\alpha : g \Rightarrow g'$ and $\beta : g' \Rightarrow g''$, for $v \in V_0$, we have

$$(\beta \circ \alpha) \cdot v = (\beta \cdot v) \diamond (\alpha \cdot v)$$

and, for $g \in G$,

$$\mathrm{id}_g \cdot v = 1_v$$

Furthermore, for $\alpha : g_1 \Rightarrow g_2$ and $\beta : g'_1 \Rightarrow g'_2$, we have

$$\begin{aligned} (\beta * \alpha) \cdot v &= (g'_2 \cdot_1 \alpha \cdot v) \diamond (\beta \cdot g_1 \cdot_0 v) \\ &= (\beta \cdot g_2 \cdot_0 v) \diamond (g'_1 \cdot_1 \alpha \cdot v) \end{aligned}$$

The last equality stems from naturality of $\beta \cdot -$.

2.3.2. From the crossed module perspective

Let us now consider the case of a 2-group presented as a group action $G \circlearrowright H$ with a morphism $t : H \to G$ satisfying

$$\begin{aligned} t(h) \cdot h' &= hh'h^{-1} \\ t(g \cdot h) &= gt(h)g^{-1} \end{aligned}$$

A 2-representation is the data of an internal category $V \coloneqq V_1 \rightrightarrows V_0$ in $\mathbf{Vect}_{\mathbb{K}}$ such that, for every $g \in G$, there is a functor $g \cdot - : V \to V$, that is, linear maps $g \cdot_0 - : V_0 \to V_0$ and $g \cdot_1 - : V_1 \to V_1$ making the following equations hold:

$$\begin{aligned} g \cdot_1 1_v &= 1_v && \text{for every } v \in V_0 \\ s(g \cdot_1 f) &= g \cdot_0 s(f) && \text{for every } f \in V_1 \\ t(g \cdot_1 f) &= g \cdot_0 t(f) && \text{for every } f \in V_1 \\ g \cdot_1 (f' \diamond f) &= (g \cdot_1 f') \diamond (g \cdot_1 f) && \text{for every } f, f' \in V_1 \text{ such that } t(f) = s(f') \end{aligned}$$

The 1-functoriality of F expresses exactly that $G \circlearrowright V_0$ and $G \circlearrowright V_1$, that is, the following equations hold:

$$\begin{aligned} e \cdot_0 v &= v && \text{for } v \in V_0 \\ e \cdot_1 f &= f && \text{for } f \in V_1 \\ (gg') \cdot_0 v &= g \cdot_0 (g' \cdot_0 v) && \text{for } v \in V_0 \\ (gg') \cdot_1 f &= g \cdot_1 (g' \cdot_1 f) && \text{for } f \in V_1 \end{aligned}$$

Let's now look at the 2-functoriality of F. For every $g \in G$ and $h \in H$, we have a linear map $h \cdot_g - : V_0 \to V_1$ satisfying the following equations

$$s\big(h \cdot_g v\big) = g \cdot_0 v$$
$$t\big(h \cdot_g v\big) = t(h)g \cdot_0 v$$
$$\big(h \cdot_g t(f)\big) \diamond (g \cdot_1 f) = (t(h)g \cdot_1 f) \diamond \big(h \cdot_g s(f)\big) \quad \text{(naturality)}$$

Furthermore, for $h, h' \in H$ and $g \in G$, and $v \in V_0$, we have

$$1_H \cdot_g v = 1_v$$
$$(h'h) \cdot_g v = h' \cdot_{t(h)g} h \cdot_g v$$

Finally, for $h, h' \in H$ and $g, g' \in G$, and $v \in V_0$, we have

$$\begin{aligned} h(g \cdot h') \cdot_{gg'} v &= \big(t(h)g \cdot_1 \big(h' \cdot_{g'} v\big)\big) \diamond \big(h \cdot_g (g' \cdot_0 v)\big) \quad (1) \\ &= \big(h \cdot_g (t(h')g' \cdot_0 v)\big) \diamond \big(g \cdot_1 \big(h' \cdot_{g'} v\big)\big) \quad (2) \end{aligned}$$

(the last equality holds by naturality of $h' \cdot_{g'} -$). Consider the group morphism

$$\begin{aligned} t : H \rtimes G &\longrightarrow G \\ (h, g) &\longmapsto t(h)g \end{aligned}$$

which is, indeed, a morphism:

$$\begin{aligned} t((h, g)(h', g')) &= t(h(g \cdot h'), gg') \\ &= t(h(g \cdot h'))gg' \\ &= t(h)t(g \cdot h')gg' \\ &= t(h)gt(h')g' \\ &= t(h, g)t((h', g')) \end{aligned}$$

The equation (1) is exactly the commutativity of the following diagram

$$
\begin{array}{ccc}
(H \rtimes G) \times (H \rtimes G) \times V_0 & \xrightarrow{\quad - \cdot - \times V_0 \quad} & (H \rtimes G) \times V_0 \\
\Big\downarrow {\scriptstyle \langle t, H \rtimes G \rangle \times \langle H \rtimes G, \pi_2 \rangle \times \Delta_{V_0}} & & \\
G \times (H \rtimes G) \times (H \rtimes G) \times G \times V_0 \times V_0 & & \\
\Big\downarrow \wr & & \Big\downarrow {\scriptstyle - \cdot -} \\
G \times (H \rtimes G) \times V_0 \times (H \rtimes G) \times G \times V_0 & & \\
\Big\downarrow {\scriptstyle G \times - \cdot - \times (H \rtimes G) \times - \cdot_0 -} & & \\
G \times V_1 \times (H \rtimes G) \times V_0 & & \\
\Big\downarrow {\scriptstyle - \cdot_1 - \times - \cdot -} & & \\
V_1 \times_{V_0} V_1 & \xrightarrow{\quad \diamond \quad} & V_1
\end{array}
$$

and the equation (2) is exactly the commutativity of the following diagram

$$
\begin{array}{ccc}
(H \rtimes G) \times (H \rtimes G) \times V_0 & \xrightarrow{\quad - \cdot - \times V_0 \quad} & (H \rtimes G) \times V_0 \\
\Big\downarrow {\scriptstyle \langle \pi_2, (H \rtimes G) \rangle \times \langle t, (H \rtimes G) \rangle \times \Delta_{V_0}} & & \\
G \times (H \rtimes G) \times G \times (H \rtimes G) \times V_0 \times V_0 & & \\
\Big\downarrow \wr & & \Big\downarrow {\scriptstyle - \cdot -} \\
(H \rtimes G) \times G \times V_0 \times G \times (H \rtimes G) \times V_0 & & \\
\Big\downarrow {\scriptstyle (H \rtimes G) \times - \cdot_0 - \times G \times - \cdot -} & & \\
(H \rtimes G) \times V_0 \times G \times V_1 & & \\
\Big\downarrow {\scriptstyle - \cdot - \times - \cdot_1 -} & & \\
V_1 \times_{V_0} V_1 & \xrightarrow{\quad \diamond \quad} & V_1
\end{array}
$$