Notes on LCCC

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1. Introduction

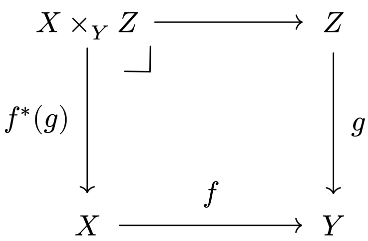
1.1. Preliminary notions

1.1.1. Change of base functor

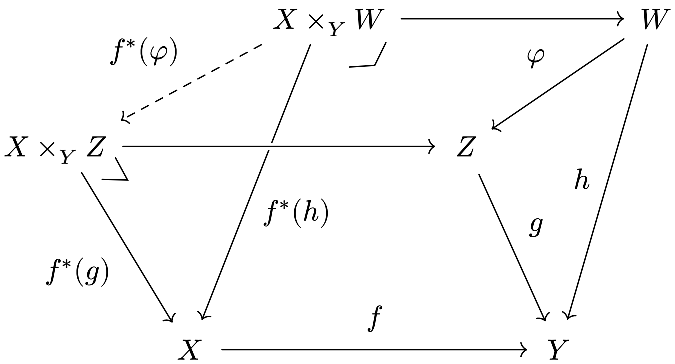
For this section, assume that we have a category $\mathcal{C}$ which has pullbacks. Let X, Y be two objects in $\mathcal{C}$, and $f : X \to Y$. We can build

$$f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$$

the "base change" functor as follows: let $g : Z \to Y$ be an element of $\mathcal{C}/Y$.



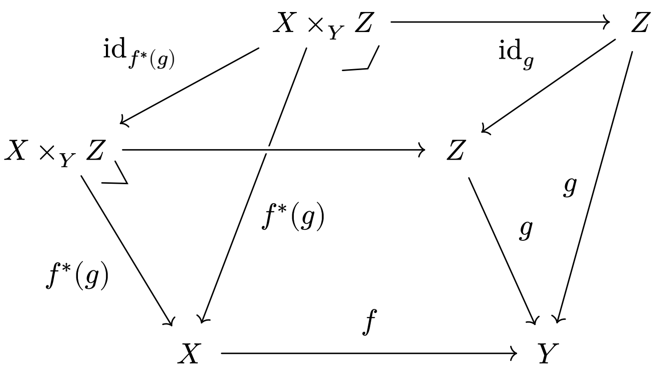
Furthermore, if $h : W \to Y$ is an other element of $\mathcal{C}/Y$ and $\varphi : h \to g$, by pullback, there exists a unique $f^*(\varphi) : X \times_Y W \to X \times_Y Z$ making the following diagram commute



hence, $f^*(\varphi)$ is a morphism $f^*(h) \longrightarrow f^*(g)$.

Lemma 1.1.1.1: f^* defines a functor.

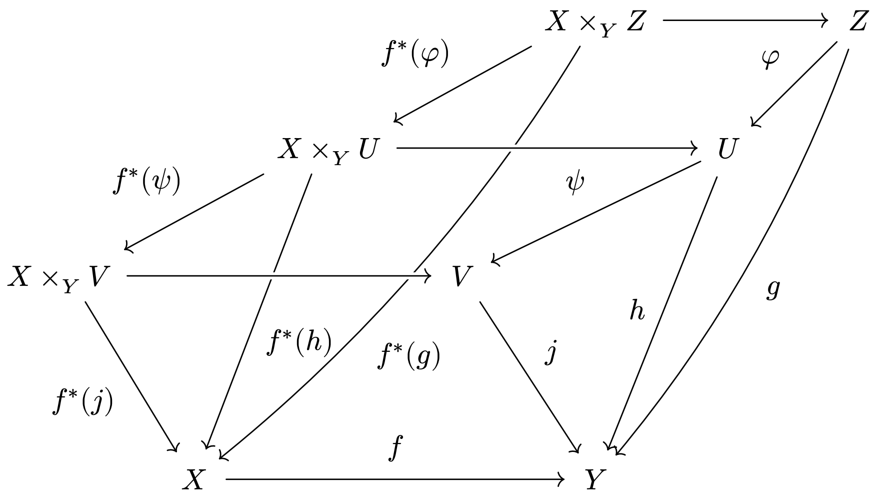
Proof: Let $(Z, g) : \mathcal{C}/Y$. Note that the following diagram commutes



thus $\mathrm{id}_{f^*(g)}$ satisfies the universaly property of $f^*(\mathrm{id}_g)$, and so

$$f^*(\mathrm{id}_g) = \mathrm{id}_{f^*(g)}$$

Let $(U, h), (V, j) : \mathcal{C}/Y$, $\varphi : (Z, g) \to (U, h)$ and $\psi : (U, h) \to (V, j)$.



Note that the topmost outer rectangle commutes because both inner square commute, and that the leftmost, outermost triangle commutes too because both inner triangle commute, making $f^*(\psi) \circ f^*(\varphi)$ satisfy the same universal condition as $f^*(\psi \circ \varphi)$, so

$$f^*(\varphi \circ \varphi) = f^*(\psi) \circ f^*(\varphi)$$

□

1.2. Main theorem

Definition 1.2.1 (Locally Cartesian Closed Category): A category $\mathcal{C}$ is *locally cartesian closed* if, for any object $X : \mathcal{C}$, the category $\mathcal{C}/X$ is cartesian closed.

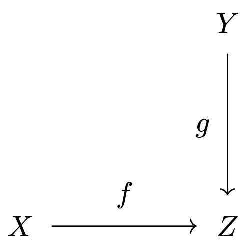
Theorem 1.2.2: A category $\mathcal{C}$ is locally cartesian closed if, and only if, it has
- pullbacks;
- for any morphism $f : X \to Y$ in $\mathcal{C}$, the functor f^* has a right adjoint Π_f, called the *dependent product at f*.

2. The direct part of the equivalence

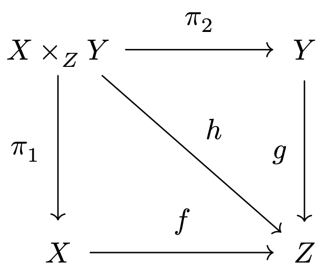
Suppose we have a locally cartesian closed category $\mathcal{C}$.

2.1. Pullbacks

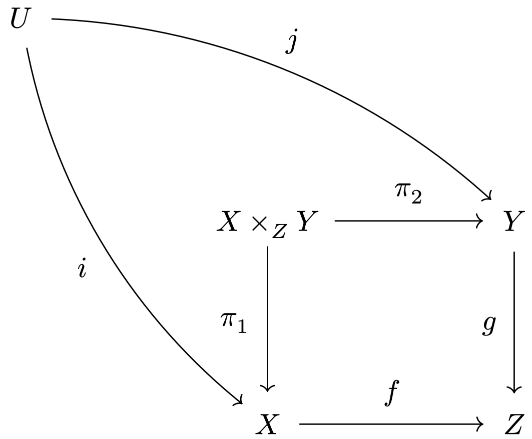
Let's show that $\mathcal{C}$ has pullbacks. Let f, g be morphisms in $\mathcal{C}$ spelling out the following diagram



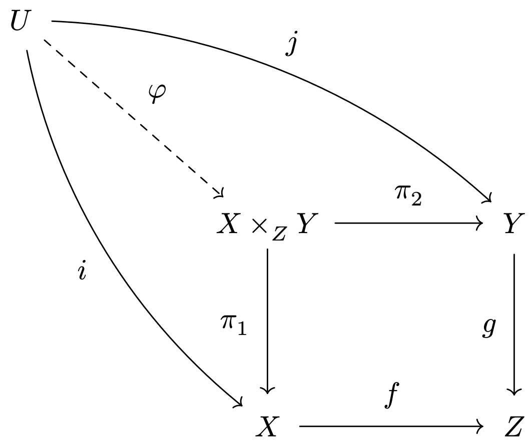
Since $\mathcal{C}/Z$ is cartesian closed, there exists a $(X \times_Z Y, h) : \mathcal{C}/Z$ the cartesian product of (X, f) and (Y, g), with projections π_1 and π_2:



This latter square (if we forget about h) is a pullback. Indeed, for any $U : \mathcal{C}$, $i : U \to X$ and $j : U \to Y$ making the following diagram commute



Let $h' = g \circ j = f \circ i$. Note that $j : (U, h') \to (Y, g)$ and $i : (U, h') \to (X, f)$ in $\mathcal{C}/Z$, so by cartesianity, there exists a unique $\varphi : (U, h') \to (X \times_Z Y, h)$ such that $\pi_2 \circ \varphi = j$ and $\pi_1 \circ \psi = i$, that is, the following diagram commutes



Note that any φ making the two triangles commute is also a morphism $(U, h') \to (X \times_Z Y, h)$ in $\mathcal{C}/Z$, so φ is indeed unique in $\mathcal{C}$.

2.2. Dependent product

2.2.1. Definition

Let $f : X \to Y$ be a morphism in $\mathcal{C}$, let us define the functor

$$\Pi_f : \mathcal{C}/X \longrightarrow \mathcal{C}/Y$$

Consider a $(Z, p) : \mathcal{C}/X$.

2.2.2. Adjunction