Notes on Grothendieck Fibrations

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Contents

1. Introduction

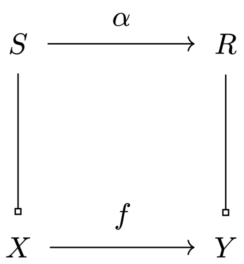
1.1. Preliminary definitions

In this section, we have two categories $\mathcal{B}$ and $\mathcal{E}$, and a functor $p : \mathcal{E} \to \mathcal{B}$.

Definition 1.1.1 (Refinement): Let $R : \mathcal{E}$ and $X : \mathcal{B}$. We say that R *refines* X, or $R \sqsubset X$, if

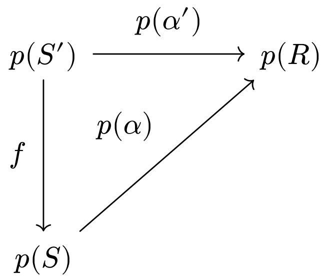
$$X = p(R)$$

We note $R \longrightarrow\!\!\!\!\!\square\; X$ to mean $R \sqsubset X$, and we say that the following diagram commutes

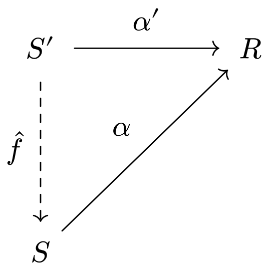


if $f = p(\alpha)$.

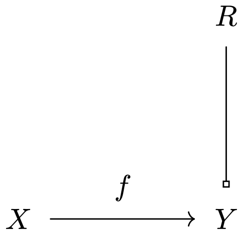
Definition 1.1.2 (Cartesian morphism): Let $R, S : \mathcal{E}$. A morphism $\alpha : S \to R$ is *cartesian* if, for any $S' : \mathcal{E}$, $\alpha' : S' \to R$, and $f : p(S') \to p(S)$ such that the following diagram commutes



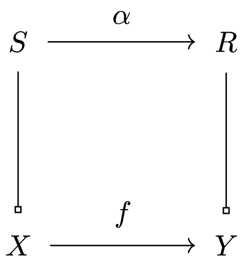
There exists a unique $\hat{f} : S' \to S$ such that $f = p\left(\hat{f}\right)$, and such that the following diagram commutes



Definition 1.1.3 (Fibration): p is said to be a *fibration* if, for any

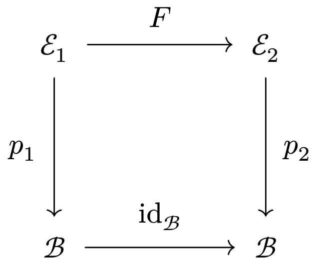


there exists a cartesion morphism α making the following commute



Definition 1.1.4 (Category of fibrations): For a base category $\mathcal{B}$, define $\mathbf{Fib}_{\mathcal{B}}$ as the category of fibrations over $\mathcal{B}$, that is, whose objects are pairs $(\mathcal{E}, p)$ with $\mathcal{E}$ a category and $p : \mathcal{E} \to \mathcal{B}$ a fibration.

Given two fibrations $p_i : \mathcal{E}_i \to \mathcal{B}$ over $\mathcal{B}$ for $i = 1, 2$, a morphism of fibrations between p_1 and p_2 is a functor $F : \mathcal{E}_1 \to \mathcal{E}_2$ making the following diagram commute



and which preserves cartesianity of morphisms.

Definition 1.1.5 (Category of pseudofunctors): For a given base category $\mathcal{B}$, define $\mathbf{Pfct}_{\mathcal{B}}$ as the category whose elements are contravariant pseudo-functors $\mathcal{P} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ in $\mathbf{Cat}$, that is,

- for each object $X : \mathcal{B}$, a category $\mathcal{P}_X$;
- for each morphism $f : X \to Y$ in $\mathcal{B}$, a functor $\mathcal{P}_f : \mathcal{P}_Y \to \mathcal{P}_X$;
- for each object $X : \mathcal{B}$, a natural isomorphism

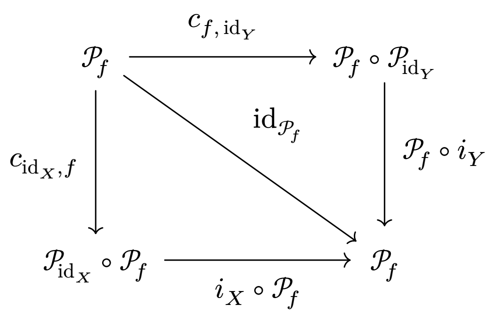
$$i_X : \mathcal{P}_{\mathrm{id}_X} \Longrightarrow \mathrm{id}_{\mathcal{P}_X}$$

  called the pseudo unit of $\mathcal{P}$ at X;
- for each morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{B}$, a natural isomorphism

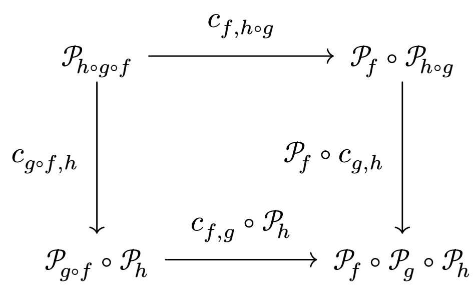
$$c_{f,g} : \mathcal{P}_{g \circ f} \Longrightarrow \mathcal{P}_f \circ \mathcal{P}_g$$

  called the pseudo composition law of $\mathcal{P}$ at f and g.

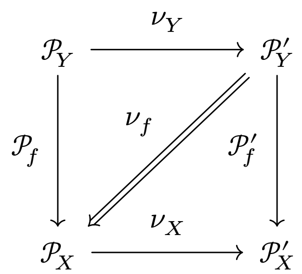
We additionally require the following coherence conditions: for $f : X \to Y$, the following diagram commutes



Furthermore, for $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, the following diagram commutes

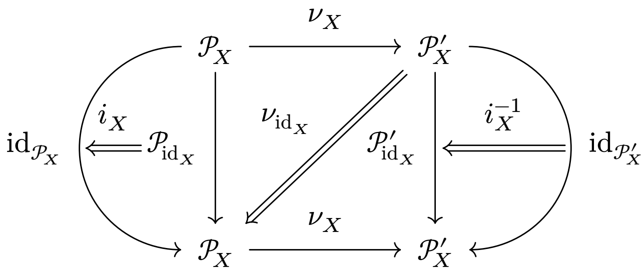


Given two pseudofunctors $\mathcal{P}$ and $\mathcal{P}'$, a morphism $\nu : \mathcal{P} \to \mathcal{P}'$ is a pseudonatural transformation between $\mathcal{P}$ and $\mathcal{P}'$, that is, for each point $X : \mathcal{B}$, a functor $\nu_X : \mathcal{P}_X \to \mathcal{P}'_X$ and, for each morphism $f : X \to Y$ in $\mathcal{B}$, a natural isomorphism



satisfying the following coherence conditions:

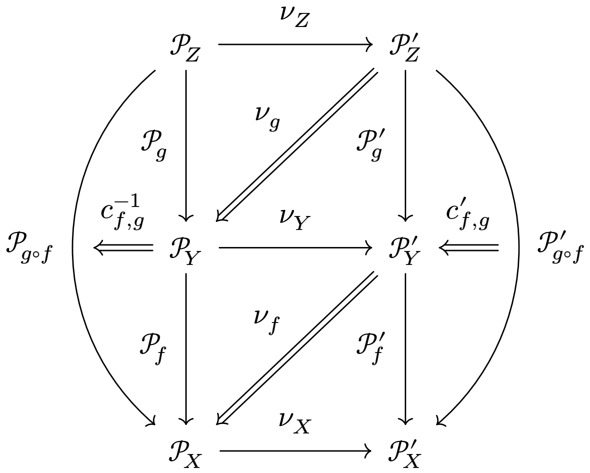
- for $X : \mathcal{B}$, the following pasting is ν_X



that is,

$$(\nu_X \circ i_X) \circ \nu_{\mathrm{id}_X} \circ (i'^{-1}_X \circ \nu_X) = \mathrm{id}_{\nu_X}$$

- if $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathcal{B}$, $\nu_{g \circ f}$ is obtained by pasting the squares (plus pseudo-composition)



that is,

$$\nu_{g \circ f} = \left(\nu_X \circ c^{-1}_{f,g}\right) \circ \left(\nu_f \circ \mathcal{P}_g\right) \circ \left(\mathcal{P}'_f \circ \nu_g\right) \circ \left(c'_{f,g} \circ \nu_Z\right)$$

1.2. Main theorem

We aim at proving the

Theorem 1.2.1 (Main theorem): For a given base category $\mathcal{B}$, we have

$$\mathbf{Fib}_{\mathcal{B}} \cong \mathbf{Pfct}_{\mathcal{B}}$$

In order to do so, we will build in Section 2 half of the equivalence, namely,

$$\Phi : \mathbf{Fib}_{\mathcal{B}} \to \mathbf{Pfct}_{\mathcal{B}}$$

and, in Section 3, the other half of the equivalence, namely,

$$\Psi : \mathbf{Pfct}_{\mathcal{B}} \to \mathbf{Fib}_{\mathcal{B}}$$

In Section 4, we will show that the two form the two halves of an equivalence, finishing the proof.

2. Fiber functor

Let's build

$$\Phi : \mathbf{Fib}_{\mathcal{B}} \to \mathbf{Pfct}_{\mathcal{B}}$$

2.1. Action of Φ on objects

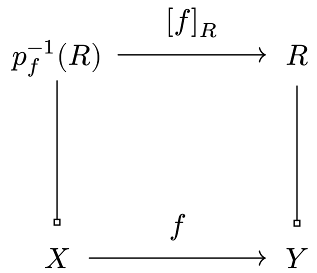
Assume we have a fibration p.

2.1.1. Definition of the fibre pseudo-functor

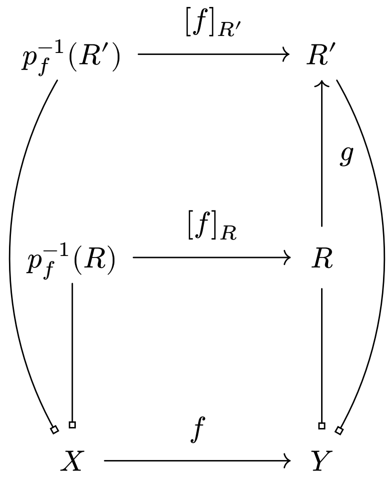
Let us build $p^{-1} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ a pseudo-functor. For $X : \mathcal{B}$,

$$p_X^{-1} = \{R : \mathcal{E} \mid R \sqsubset X\}$$
$$p_X^{-1}(S, R) = \{\alpha : S \to R \mid p(\alpha) = \mathrm{id}_X\}$$

Let $X, Y : \mathcal{B}$ and $f : X \to Y$. Let's define $p_f^{-1} : p_Y^{-1} \to p_X^{-1}$ by noticing that, for each $R : p_Y^{-1}$, by the fibration condition on p, there exists a cartesian morphism $[f]_R$



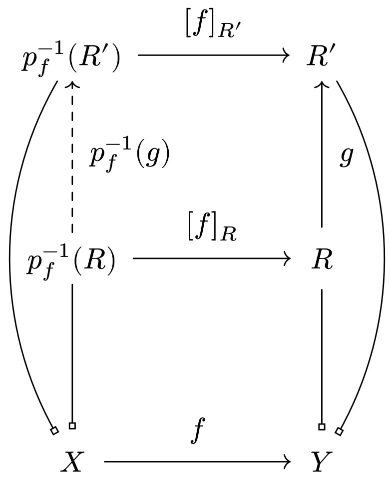
Furthermore, for $R, R' : p_Y^{-1}$ and $g : R \to R'$, we have the following diagram



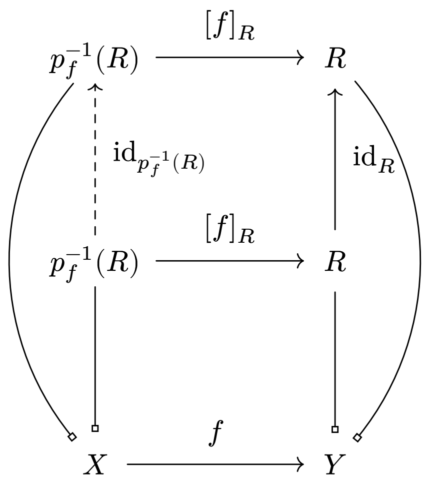
By cartesianity of $\iota_{R'}$, there exists a unique $p_f^{-1}(g) : p_f^{-1}(R) \to p_f^{-1}(R')$ st

$$p\Big(p_f^{-1}(g)\Big) = \mathrm{id}_X$$
$$[f]_{R'} \circ p_f^{-1}(g) = g \circ [f]_R$$

ie. the following diagram commutes



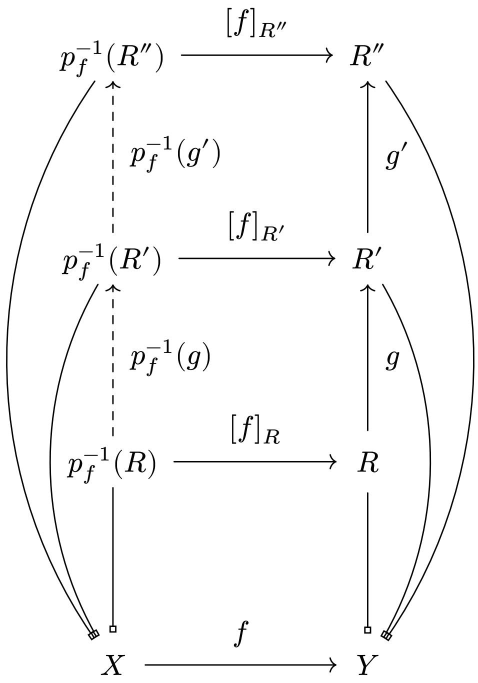
Let us indeed check that this defines a functor. For any $R : p_Y^{-1}$, note that



$\mathrm{id}_{p_f^{-1}(R)}$ satisfies the universal property of $p_f^{-1}(\mathrm{id}_R)$, so we have

$$p_f^{-1}(\mathrm{id}_R) = \mathrm{id}_{p_f^{-1}(R)}$$

Let now $R, R', R'' : p_Y^{-1}$, $g : R \to R'$ and $g' : R' \to R''$.



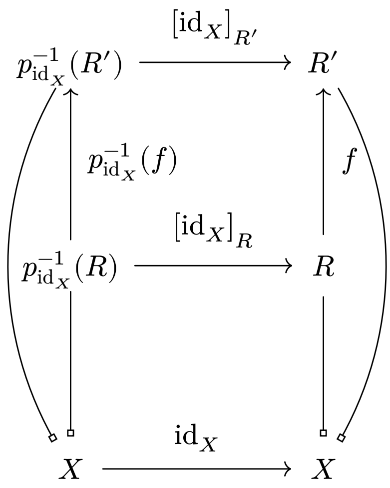
Note that $p_f^{-1}(g') \circ p_f^{-1}(g)$ satisfies the universal property of $p_f^{-1}(g' \circ g)$, so we have

$$p_f^{-1}(g' \circ g) = p_f^{-1}(g') \circ p_f^{-1}(g)$$

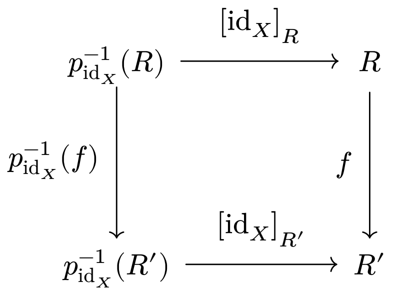
Let us now show that p^{-1} indeed defines a pseudo-functor.

2.1.2. Pseudo identity law

For $X : \mathcal{B}^{\mathrm{op}}$, let us first exhibit a natural isomorphism $p_{\mathrm{id}_X}^{-1} \tilde{\Rightarrow} \mathrm{id}_X$. For $R : p_X^{-1}$, we have $[\mathrm{id}_X]_R : p_{\mathrm{id}_X}^{-1}(R) \to \mathrm{id}_X(R)$. This defines a natural transformation. Indeed, for $R, R' : p_X^{-1}$ and $f : R \to R'$, the following diagram commutes by definition of $p_{\mathrm{id}_X}^{-1}(f)$:

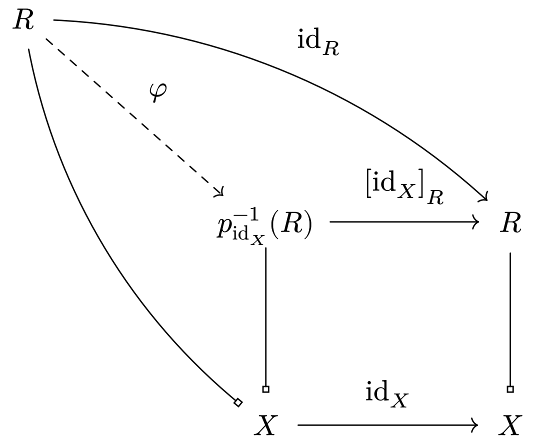


So in particular the upper square commutes



hence $[\mathrm{id}_X]$ is natural. Let's show that each component is an isomorphism.

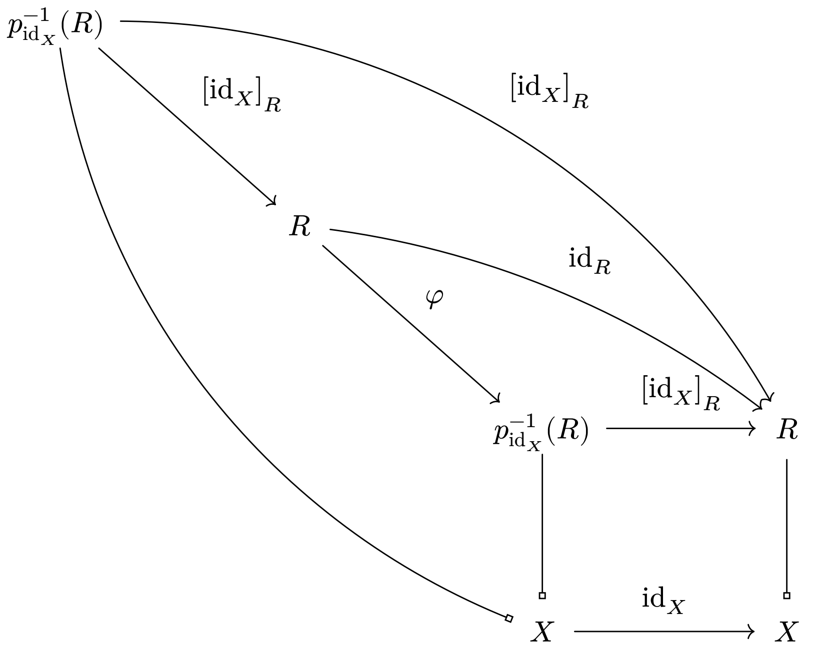
There is a unique morphism $\varphi : R \to p_{\mathrm{id}_X}^{-1}(R)$ making the following diagram commute



So

$$[\mathrm{id}_X]_R \circ \varphi = \mathrm{id}_R$$

Furthermore, the following diagram commutes



Meaning that $\varphi \circ [\mathrm{id}_X]_R$ satisfies the universal property of $[\mathrm{id}_X]_R$ with respect to $[\mathrm{id}_X]_R$. But so does the identity, so, by unicity, we have

$$\varphi \circ [\mathrm{id}_X]_R = \mathrm{id}_{p_{\mathrm{id}_X}^{-1}(R)}$$

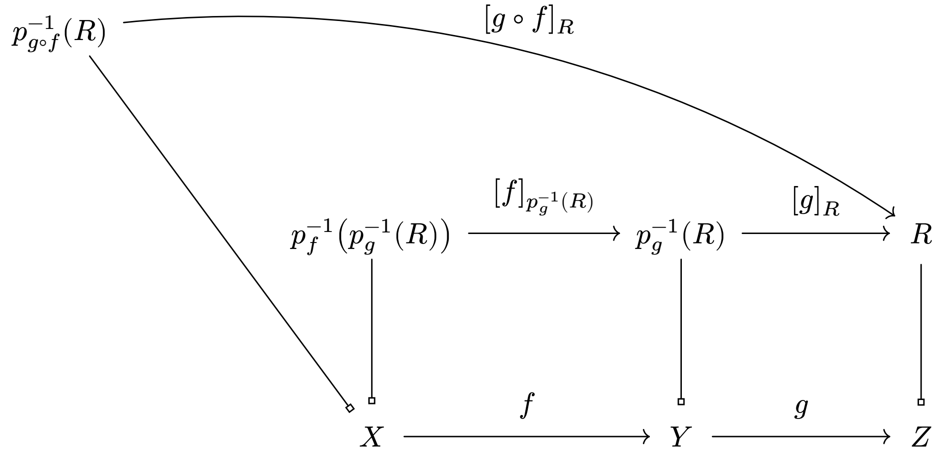
Hence $[\mathrm{id}_X]_R$ is an iso.

2.1.3. Pseudo-composition law

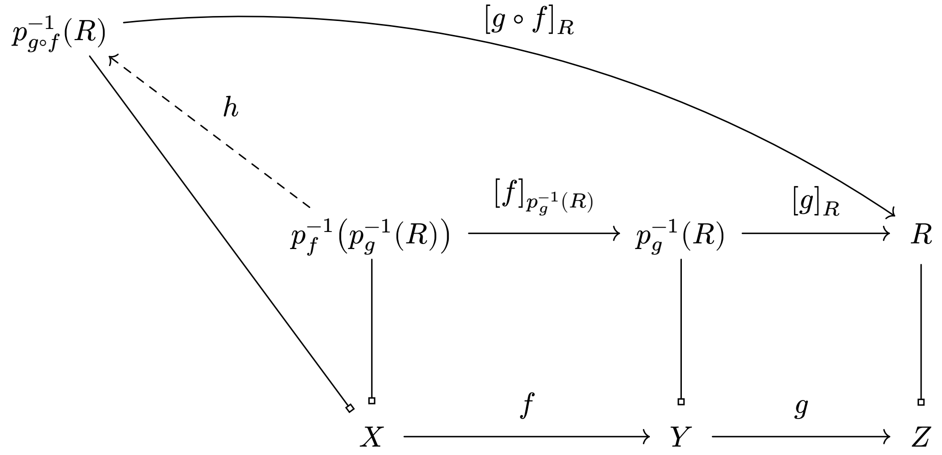
Lemma 2.1.3.1 (Pseudo-composition law): Let $X, Y, Z : \mathcal{B}$, and $f : X \to Y$, $g : Y \to Z$. There is a natural isomorphism

$$[f, g] : p_{g \circ f}^{-1} \Longrightarrow p_f^{-1} \circ p_g^{-1}$$

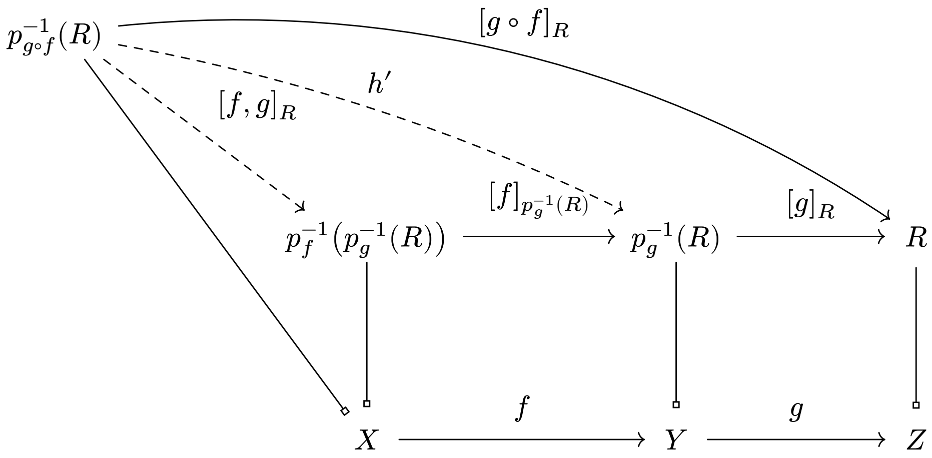
Proof: Let $R : p_Z^{-1}$, and consider the following diagram



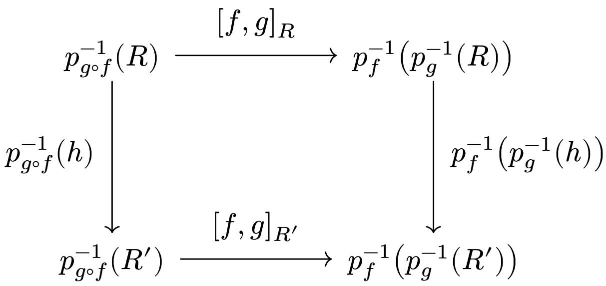
The fact that $[g \circ f]_R$ is cartesian gives a unique morphism $h : p_f^{-1}(p_g^{-1}(R)) \to p_{g\circ f}^{-1}(R)$ making the diagram commute:



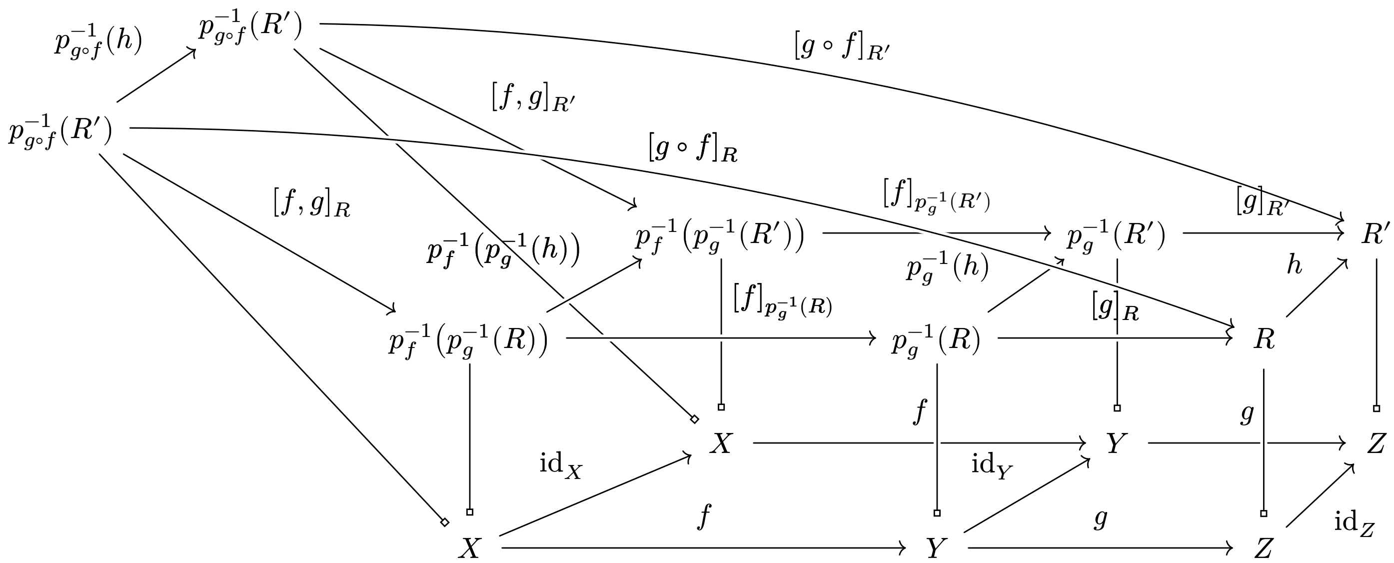
Conversely, the cartesianity of $[g]_R$, and then $[f]_{p_g^{-1}(R)}$ gives $h' : p_{g\circ f}^{-1}(R) \to p_g^{-1}(R)$, then $[f, g]_R : p_{g\circ f}^{-1}(R) \to p_f^{-1}(p_g^{-1}(R))$ making the following commute



In particular, $[f,g]_R$ and h must be each other's inverse. We have to show that this construction is natural. Let $R, R' : p_Z^{-1}$ and $h : R \to R'$. We want to show that the following diagram commutes



Note that in the following diagram



$p_{g\circ f}^{-1}(h)$ is the unique solution to the universal problem of living in the fiber above X and making the top-most square commute. Hence, to prove that

$$[f,g]_{R'} \circ p_{g\circ f}^{-1}(h) = p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R$$

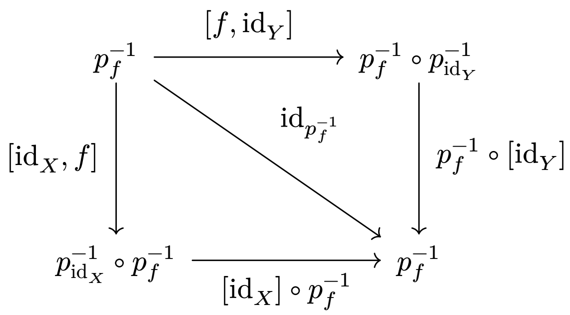
it suffices to show that $[f,g]_{R'}^{-1} \circ p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R$ also satisfies this universal property. Each of these three morphisms lives in the fiber above X, so so does their composition. Furthermore,

$$
\begin{aligned}
[g \circ f]_{R'} \circ [f,g]_{R'}^{-1} \circ p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R &= [g]_{R'} \circ [f]_{p_g^{-1}(R')} \circ p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R && \text{by definition of } [f,g]_{R'} \\
&= [g]_{R'} \circ p_g^{-1}(h) \circ [f]_{p_g^{-1}(R)} \circ [f,g]_R && \text{by definition of } p_f^{-1}\big(p_g^{-1}(h)\big) \\
&= h \circ [g]_R \circ [f]_{p_g^{-1}(R)} \circ [f,g]_R && \text{by definition of } p_g^{-1}(h) \\
&= h \circ [g \circ f]_R && \text{by definition of } [f,g]_R
\end{aligned}
$$

□

2.1.4. Identity/composition coherence

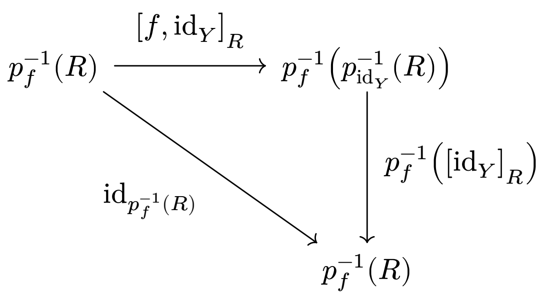
Let $X, Y : \mathcal{B}$ and $f : X \to Y$ We have to check that the following diagram commutes



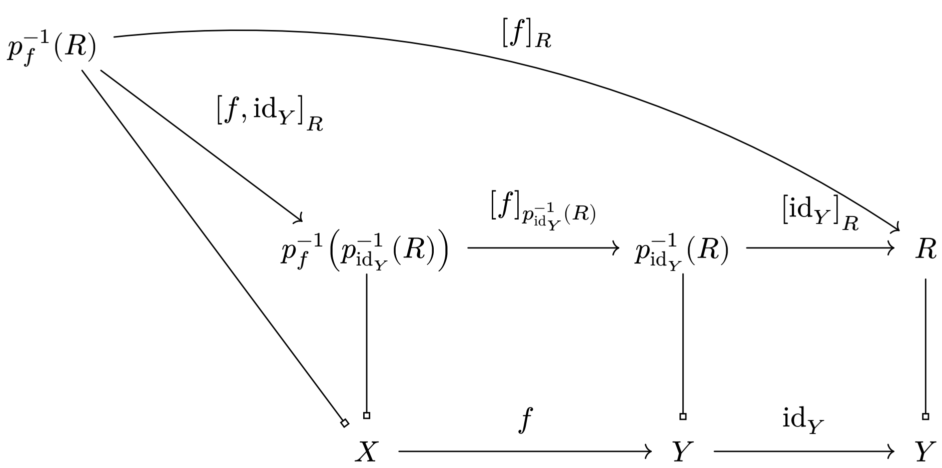
Let's show that each triangle commutes independently.

2.1.4.1. Upper triangle

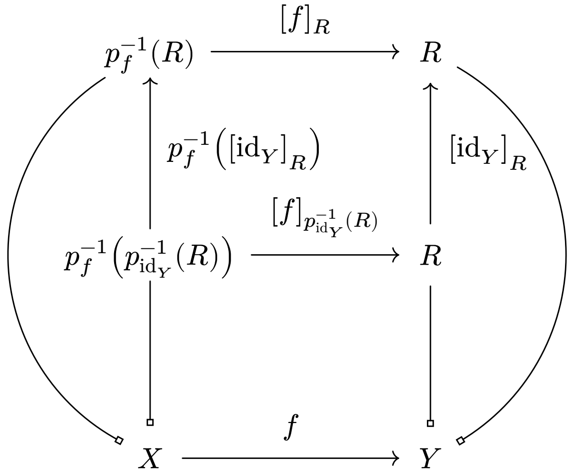
Let $R : p_Y^{-1}$. We have to check the commutation of the following diagram



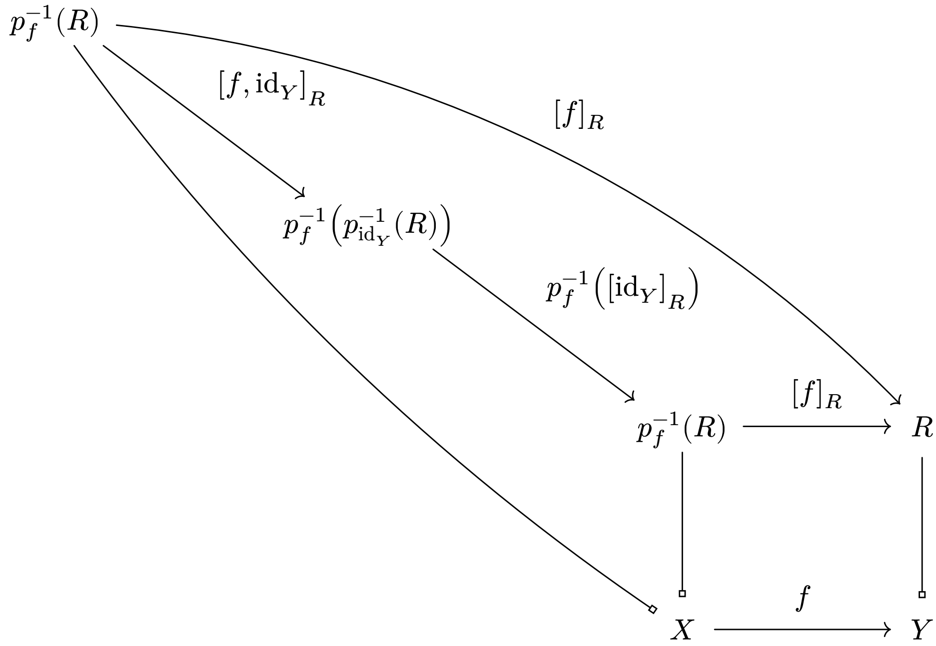
By definition of $[f, \mathrm{id}_Y]$, the following diagram commutes



and we also have



Hence, by stitching the two together, we have that the following diagram commutes

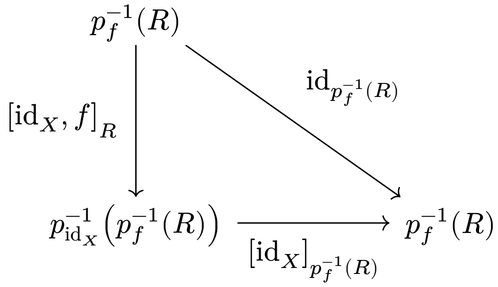


By cartesianity of $[f]_R$, $p_f^{-1}\left([\mathrm{id}_Y]_R\right) \circ [f, \mathrm{id}_Y]_R$ is unique making this diagram commute; but since so does $\mathrm{id}_{p_f^{-1}(R)}$, we must have

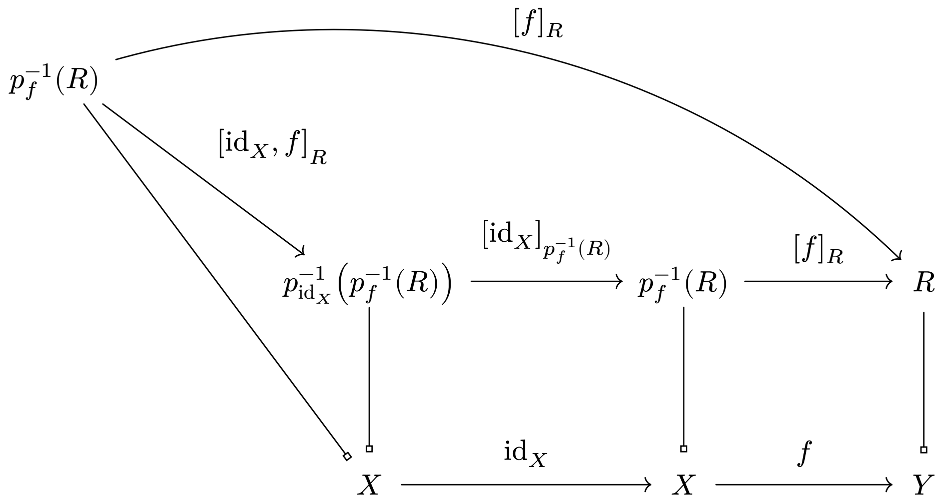
$$p_f^{-1}\left([\mathrm{id}_Y]_R\right) \circ [f, \mathrm{id}_Y]_R = \mathrm{id}_{p_f^{-1}(R)}$$

2.1.4.2. Lower triangle

Let $R : p_Y^{-1}$. We have to show the commutation of the following diagram



By definition of $[\mathrm{id}_X, f]$, the following diagram commutes



but, by cartesianity of $[f]_R$, $[\mathrm{id}_X]_{p_f^{-1}(R)} \circ [\mathrm{id}_X, f]_R$ is unique making this diagram commute. Because $\mathrm{id}_{p_f^{-1}(R)}$ also makes it commute, we must have

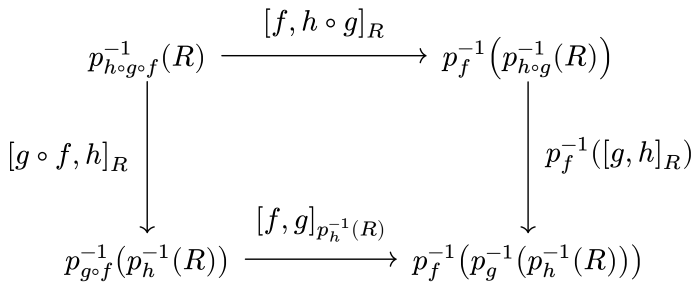
$$[\mathrm{id}_X]_{p_f^{-1}(R)} \circ [\mathrm{id}_X, f]_R = \mathrm{id}_{p_f^{-1}(R)}$$

2.1.5. Composition/composition coherence

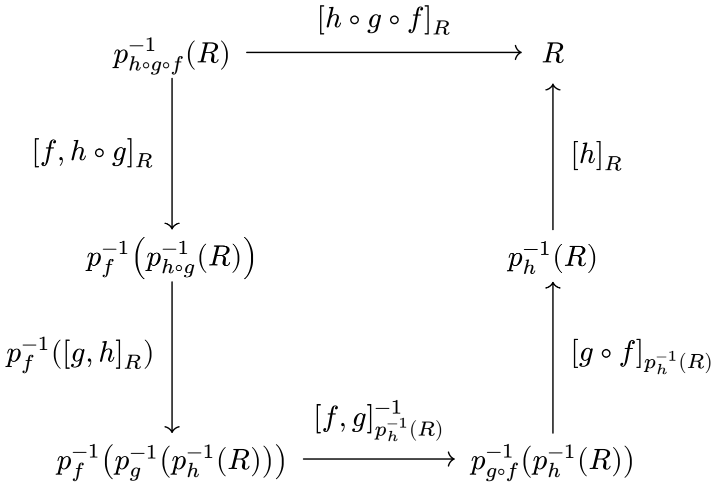
Let $W, X, Y, Z : \mathcal{B}$ and

$$W \xrightarrow{\quad f \quad} X \xrightarrow{\quad g \quad} Y \xrightarrow{\quad h \quad} Z$$

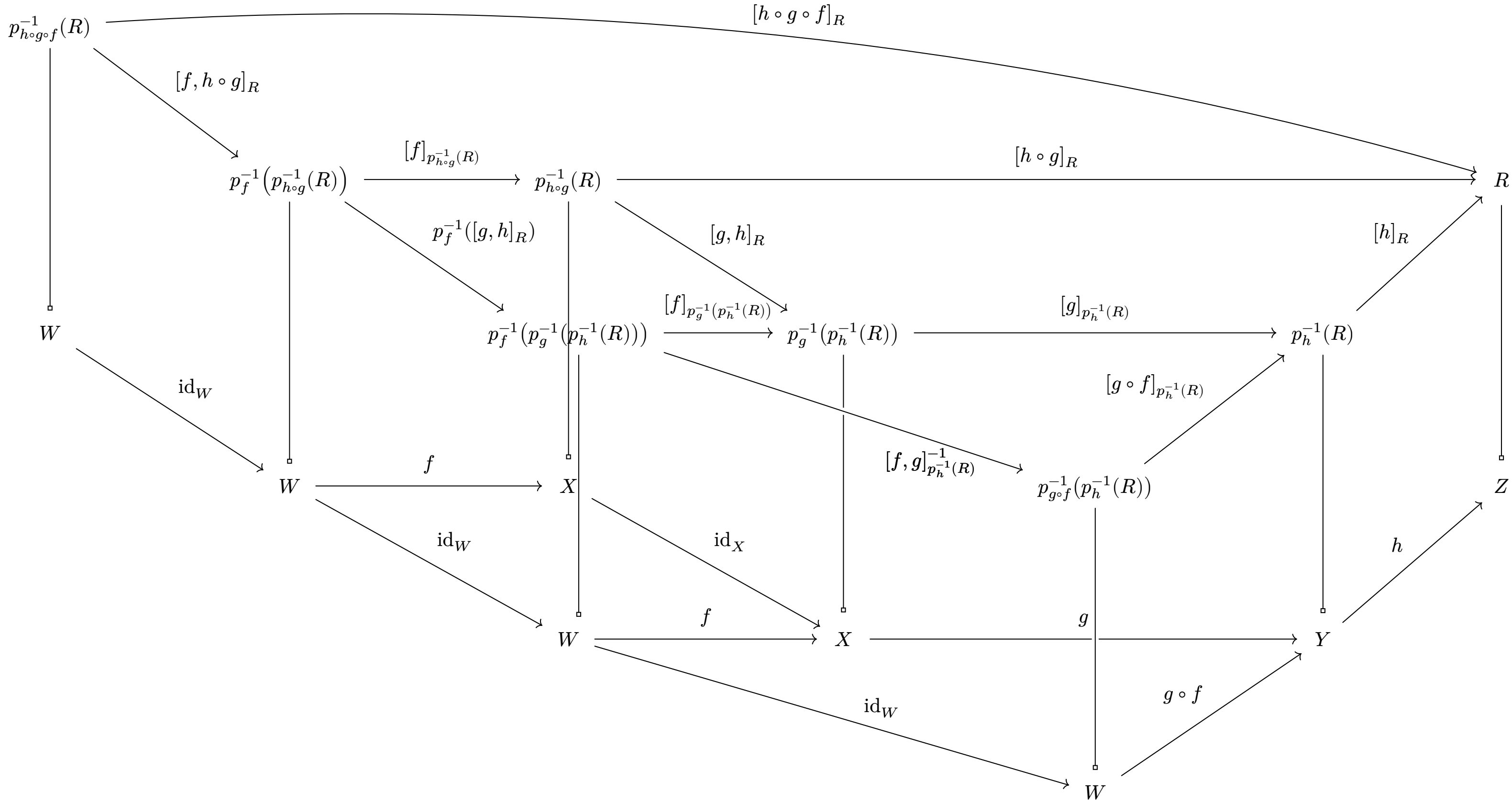
Let $R : p_Z^{-1}$, we want to show that the following diagram commutes



It suffices to show that $[f, g]_{p_h^{-1}(R)}^{-1} \circ p_f^{-1}([g, h]_R) \circ [f, h \circ g]_R$ satisfies the universaly property of $[g \circ f, h]_R$, that is, the following diagram commutes



In the following diagram, each inner diagram commutes, hence the outermost diagram commutes



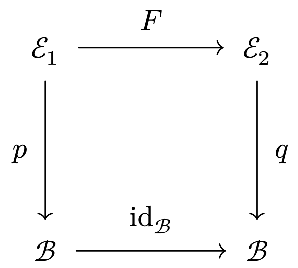
which is exactly what we wanted.

We can therefore define

$$\Phi(p) = p^{-1}$$

2.2. Action of Φ on morphisms

Let $p : \mathcal{E}_1 \to \mathcal{B}$ and $q : \mathcal{E}_2 \to \mathcal{B}$ be two fibrations, and $F : p \to q$ be a morphism



We want to define $\nu^F : p^{-1} \to q^{-1}$. Let $X : \mathcal{B}$,

$$\begin{aligned} \nu^F_X : p^{-1}(X) &\longrightarrow q^{-1}(X) \\ S &\longmapsto F(S) \\ f &\longmapsto F(f) \end{aligned}$$

Which is well defined because, if $p(S) = X$, then

$$q(F(S)) = p(S) = X$$

and if $f : S \to R$ is in the fiber above X, then

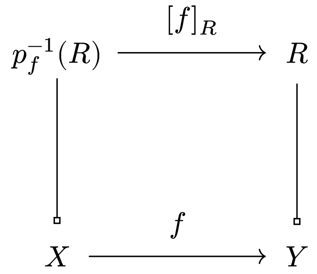
$$q(F(f)) = p(f) = \mathrm{id}_X$$

so $F(f)$ also lives in the fiber above X. ν_X^F is clearly functorial, because F is.

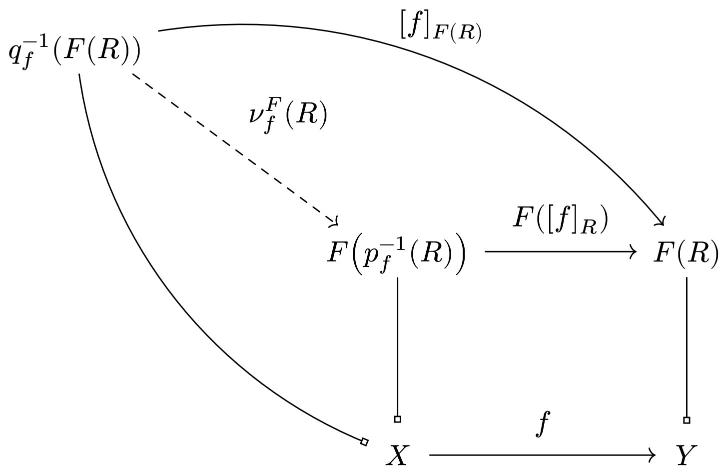
Now, let $f : X \to Y$ in $\mathcal{B}$

$$\nu_f^F(R) : q_f^{-1}(F(R)) \longrightarrow F\left(p_f^{-1}(R)\right)$$

is defined by noting that we have the following commuting diagram



and so, by cartesianity of $F([f]_R)$, which stems from that of $[f]_R$ because F preserves cartesianity,

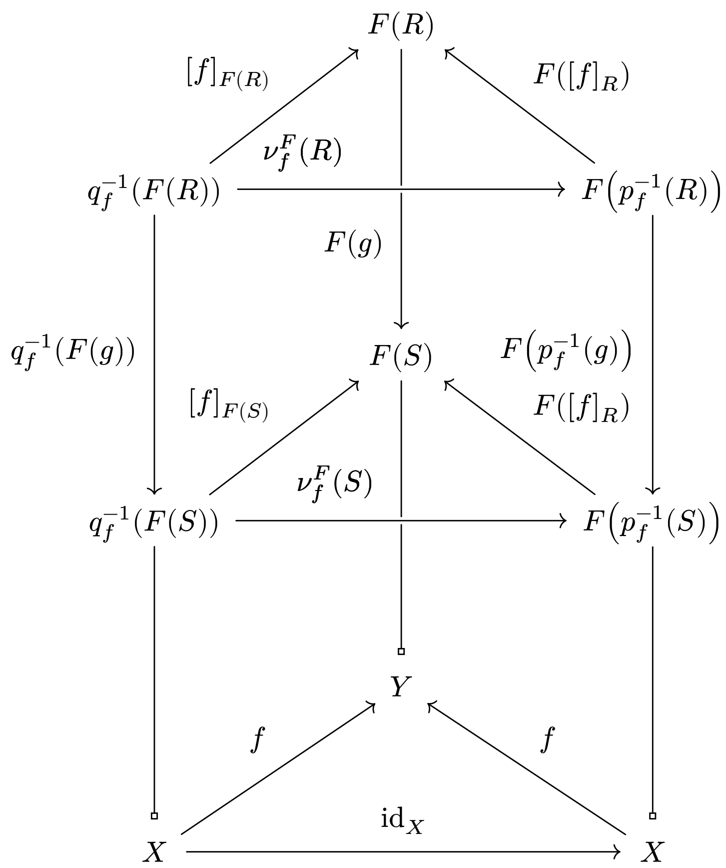


2.2.1. ν_f^F is an isomorphism

2.2.1.1. Naturality

Lemma 2.2.1.1.1: ν_f^F is a natural transformation.

Proof: Let $g : R \to S$ be a morphism in p_Y^{-1}.



We have to show that the upper front square commutes. This stems from the fact that $q_f^{-1}(F(g))$ has the universal property of living in the fiber over X, and making the left-most square commute, so we just need to check that the same is true for

$$\nu_f^F(S)^{-1} \circ F\left(p_f^{-1}(g)\right) \circ \nu_f^F(R)$$

which is true because the two triangles and the right-most square commute in the above diagram. □

2.2.1.2. Coherences

Lemma 2.2.1.2.1: ν^F is a morphism.

Proof: We have shown that, for any f, ν_f^F is a natural transformation. We just have to check that ν^F satisfies the coherence conditions.

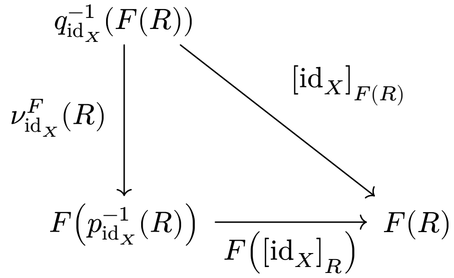
- Let $X : \mathcal{B}$. Let $R : p_X^{-1}$. We have to check that

  $$\mathrm{id}_{\nu_X(R)} = \left(\nu_X^F\left([\mathrm{id}_X]_R\right)\right) \circ \nu_{\mathrm{id}_X}^F(R) \circ [\mathrm{id}_X]_{\nu_X(R)}^{-1}$$

  that is,

  $$[\mathrm{id}_X]_{F(R)} = F\left([\mathrm{id}_X]_R\right) \circ \nu_{\mathrm{id}_X}^F(R)$$

  which is, in diagrammatic form,

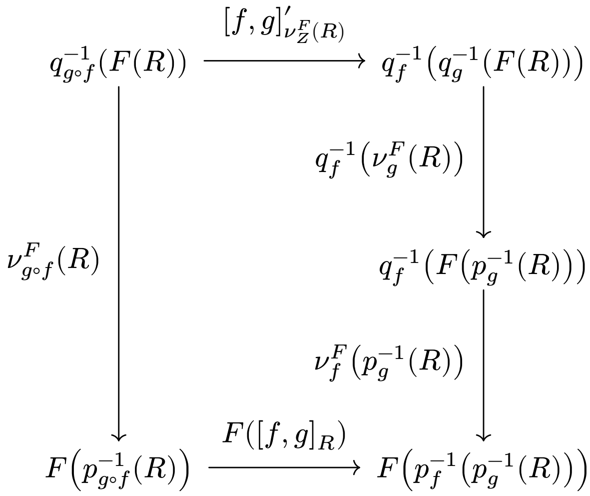


the commutation of this diagram is exactly the definition of $\nu_{\mathrm{id}_X}^F(R)$.

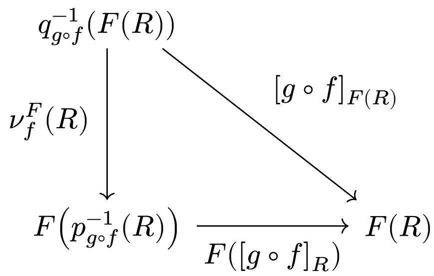
- Let $X, Y, Z : \mathcal{B}$ be three objects, $f : X \to Y$ and $g : Y \to Z$ be two morphisms in $\mathcal{B}$. Let $R : p_Z^{-1}$. We have to check that

$$\nu_{g\circ f}^F(R) = \nu_X^F([f,g]_R^{-1}) \circ \nu_f^F\big(p_g^{-1}(R)\big) \circ q_f^{-1}\big(\nu_g^F(R)\big) \circ [f,g]'_{\nu_Z^F(R)}$$

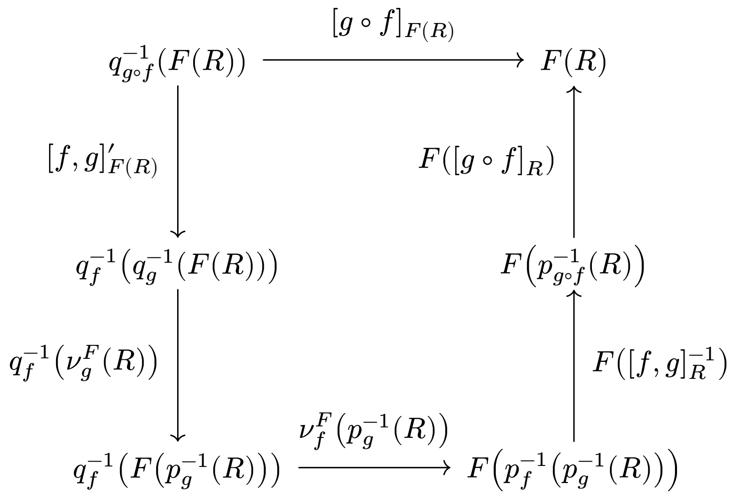
that is, that the following diagram commutes



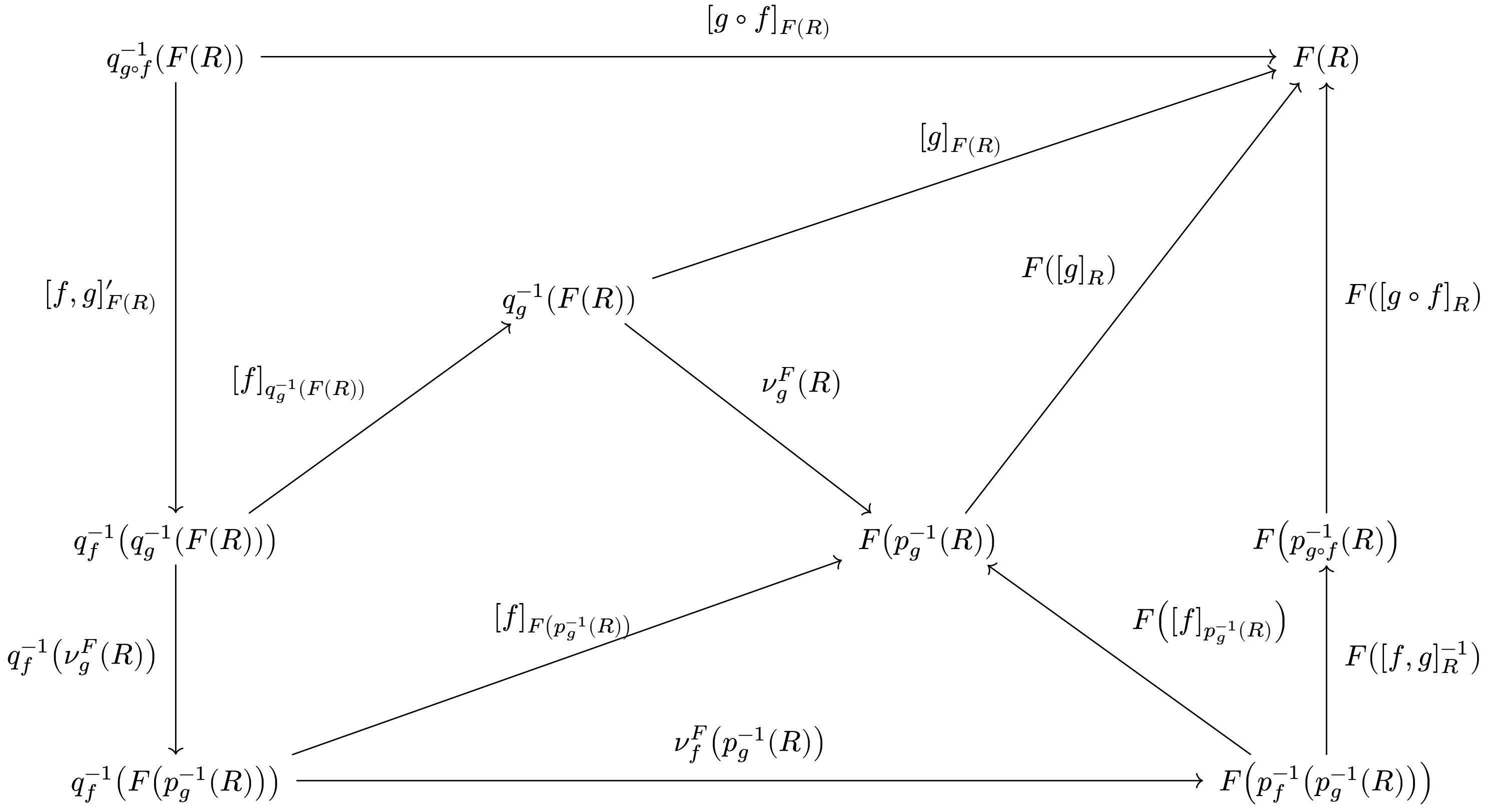
$\nu_{g\circ f}^F(R)$ is defined as the unique map in the fiber above X that makes the following diagram commute



Hence, we just need to show that the following diagram commutes



Indeed, we can fill it with commuting diagrams as follows



□

2.2.1.3. Iso

Lemma 2.2.1.3.1: $\nu_f^F(R)$ is an isomorphism.

Proof: This stems from the fact that $[f]_{F(R)}$ is cartesian. □

We therefore define

$$\Phi(F) = \nu^F$$

3. Grothendieck construction

In this section, we will define a functor $\Psi : \mathbf{Pfct}_{\mathcal{B}} \to \mathbf{Fib}_{\mathcal{B}}$.

3.1. Action of Ψ on objects

Let $\mathcal{P} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ be a pseudo-functor. Let's build a fibration over B out of it.

Definition 3.1.1 (Total category): The total category $\int \mathcal{P}$ has

- objects: pairs (X, x) with $X : \mathcal{B}^{\mathsf{op}}$ and $x : \mathcal{P}_X$;
- morphisms between two objects (A, a) and (B, b): pairs (f_1, f_2) with $f_1 : A \to B$ in $\mathcal{B}$ and $f_2 : a \to \mathcal{P}_{f_1}(b)$.
- identities for $(X, x) : \int \mathcal{P}$: $\left(\mathrm{id}_{(X,x)}\right)_0 = \mathrm{id}_X$ and

$$\left(\mathrm{id}_{(X,x)}\right)_1 : x \longrightarrow \mathcal{P}_{\mathrm{id}_X}(x)$$

$$\left(\mathrm{id}_{(X,x)}\right)_1 = i_X^{-1}(x)$$

- composition, given $(A, a), (B, b), (C, c) : \int \mathcal{P}$, $(f_1, f_2) : (A, a) \to (B, b)$ and $(g_1, g_2) : (B, b) \to (C, c)$: $(h_1, h_2) = (g_1, g_2) \circ (f_1, f_2)$ by
  - $h_1 : A \to C = g_1 \circ f_1$
  - $h_2 : a \to \mathcal{P}_{g_1 \circ f_1}(c)$ by

$$a \xrightarrow{f_2} \mathcal{P}_{f_1}(b) \xrightarrow{\mathcal{P}_{f_1}(g_2)} \mathcal{P}_{f_1}\left(\mathcal{P}_{g_1}(c)\right) \xrightarrow{c^{-1}_{f_1,g_1}(c)} \mathcal{P}_{g_1 \circ f_1}(c)$$

Lemma 3.1.1: Let f be an isomorphism in $\int \mathcal{P}$. f_1 and f_2 are invertible.

Proof: Let $(f_1, f_2) : (A, a) \to (B, b)$ a morphism in $\int \mathcal{P}$, and $(g_1, g_2) : (B, b) \to (A, a)$ such that

$$(f_1, f_2) \circ (g_1, g_2) = \mathrm{id}_{B,b}$$

$$(g_1, g_2) \circ (f_1, f_2) = \mathrm{id}_{A,a}$$

We have that $f_1 \circ g_1 = \mathrm{id}_B$ and $g_1 \circ f_1 = \mathrm{id}_A$, so f_1 is invertible and $f_1^{-1} = g_1$.

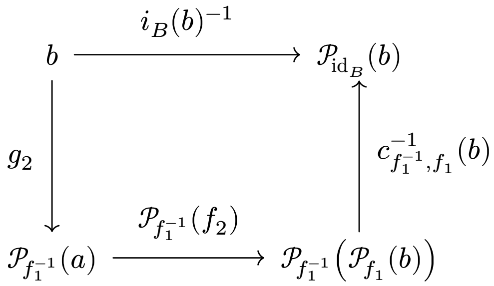
Furthermore, the following diagram commute

$$\begin{array}{ccc}
a & \xrightarrow{i_A(a)^{-1}} & \mathcal{P}_{\mathrm{id}_A}(a) \\
{\scriptstyle f_2}\big\downarrow & & \big\uparrow{\scriptstyle c_{f_1, f_1^{-1}}(a)^{-1}} \\
\mathcal{P}_{f_1}(b) & \xrightarrow{\mathcal{P}_{f_1}(g_2)} & \mathcal{P}_{f_1}\left(\mathcal{P}_{f_1^{-1}}(a)\right)
\end{array}$$

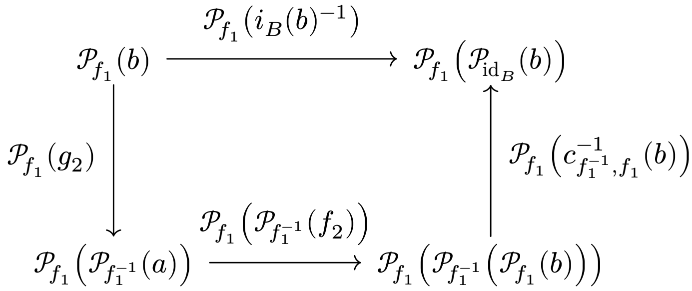
So we have a candidate for the inverse of f_2, namely,

$$\hat{f}_2 := i_A(a) \circ c^{-1}_{f_1, f_1^{-1}}(a) \circ \mathcal{P}_{f_1}(g_2)$$

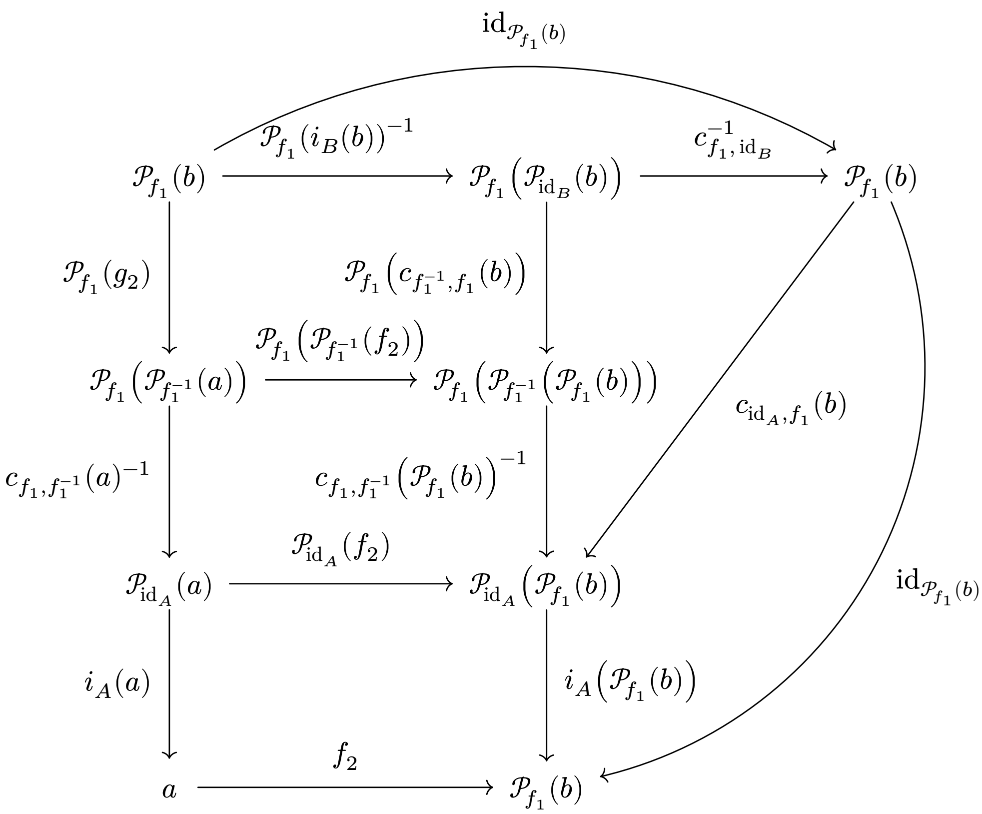
because the diagram above states that $\hat{f}_2 \circ f_2 = \mathrm{id}_a$. Furthermore, the following diagram commutes



Hence so does its image by $\mathcal{P}_{f_1}$



Thus the following diagram commutes (the other inner squares/triangles are coherence conditions)



the outermost diagram states precisely

$$f_2 \circ \hat{f}_2 = \mathrm{id}_{\mathcal{P}_{f_1}(b)}$$

□

Definition 3.1.2 (Forgetful fibration): We can now define the forgetful fibration

$$\begin{aligned} \pi(\mathcal{P}) : \quad \int \mathcal{P} &\longrightarrow \mathcal{B} \\ (A, a) &\longmapsto A \\ (f_1, f_2) &\longmapsto f_1 \end{aligned}$$

which is clearly functorial.

Lemma 3.1.2: The forgetful fibration is a fibration.

Proof: Let $A, B : \mathcal{B}$, $f : A \to B$ and $b : \mathcal{P}_B$, ie we have the following diagram:

$$\begin{array}{ccc} & & (B, b) \\ & & \big| \\ A & \xrightarrow{f} & B \end{array}$$

We can lift f as

$$\begin{array}{ccc} \big(A, \mathcal{P}_f(b)\big) & \xrightarrow{\left(f, \mathrm{id}_{\mathcal{P}_f(b)}\right)} & (B, b) \\ \big| & & \big| \\ A & \xrightarrow{f} & B \end{array}$$

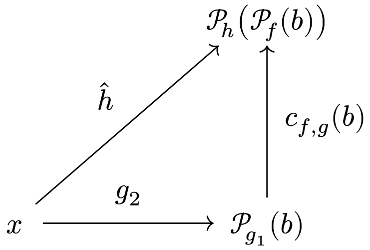
Hence, we just have to show that $\left(f, \mathrm{id}_{\mathcal{P}_f(b)}\right)$ is cartesian. Let $X : \mathcal{B}$, $x : X$, $(g_1, g_2) : (X, x) \to (B, b)$ and $h : X \to A$ such that $g_1 = f \circ h$:

$$\begin{array}{ccc} X & & \\ {\scriptstyle h}\downarrow & \searrow^{g_1} & \\ A & \xrightarrow{f} & B \end{array}$$

We have to show there is a unique $\hat{h} : x \to \mathcal{P}_h\big(\mathcal{P}_f(b)\big)$ with

$$\begin{array}{ccc} x & \xrightarrow{g_2} & \mathcal{P}_{g_1}(b) \\ {\scriptstyle \hat{h}}\downarrow & & \uparrow{\scriptstyle c_{f,g}^{-1}(b)} \\ \mathcal{P}_h\big(\mathcal{P}_f(b)\big) & \xrightarrow{\mathcal{P}_h\left(\mathrm{id}_{\mathcal{P}_f(b)}\right)} & \mathcal{P}_f(\mathcal{P}_h(b)) \end{array}$$

which is equivalent to the following diagram commuting



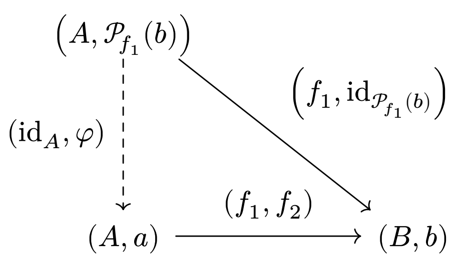
but it is obvious that there is exactly one $\hat{h}$ that makes this commute, namely,

$$\hat{h} = c_{f,g}(b) \circ g_2$$

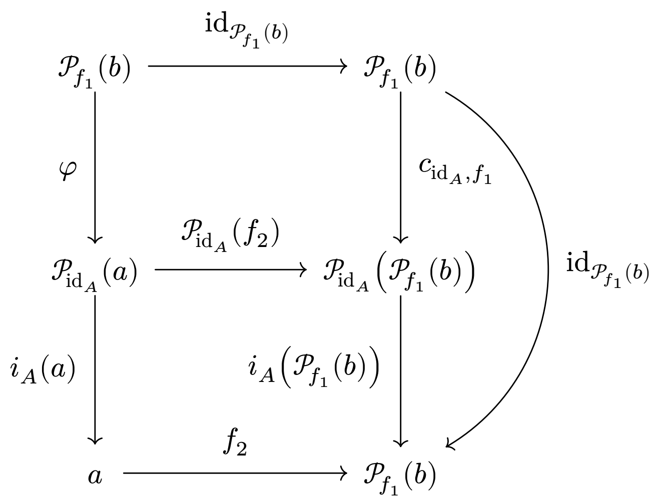
□

Lemma 3.1.3: Let (f_1, f_2) be a cartesian morphism in $\int \mathcal{P}$. f_2 is an isomorphism.

Proof: Let $(f_1, f_2) : (A, a) \to (B, b)$ be a cartesian morphism. In the previous proof, we have established that $\left(f_1, \mathrm{id}_{\mathcal{P}_{f_1}(b)}\right)$ is cartesian. Hence, there exists a unique isomorphism (id_A, φ) making the following diagram commute



hence the top square of this diagram commutes



the lower square commutes by naturality of i_A, and the triangle commutes by a coherence condition. Therefore,

$$f_2 \circ i_A(a) \circ \varphi = \mathrm{id}_{\mathcal{P}_{f_1}(b)}$$

By Lemma 3.1.1, (id_A, φ) is an isomorphism, so φ is too and hence

$$f_2 = (i_A(a) \circ \varphi)^{-1}$$

so f_2 is an isomorphism. □

Lemma 3.1.4: Let (f_1, f_2) be a morphism in $\int \mathcal{P}$, with f_2 an isomorphism. (f_1, f_2) is cartesian.

Proof: Let $(f_1, f_2) : (X, x) \to (Y, y)$, with $f_2 : x \to \mathcal{P}_{f_1}(y)$ an isomorphism, $(g_1, g_2) : (Z, z) \to (Y, y)$ and $h : Z \to X$ such that $g_1 = f_1 \circ h$. We want to find a unique $\hat{h} : z \to \mathcal{P}_h(x)$ such that $(g_1, g_2) = (f_1, f_2) \circ \left(h, \hat{h}\right)$, which is equivalent to the commutation of the following diagram

$$\begin{array}{ccc} z & \xrightarrow{\quad g_2 \quad} & \mathcal{P}_{g_1}(y) \\ {\scriptstyle \hat{h}} \Big\downarrow \vdots & & \Big\downarrow {\scriptstyle c_{h,f_1}(y)} \\ \mathcal{P}_h(x) & \xrightarrow{\quad \mathcal{P}_h(f_2) \quad} & \mathcal{P}_h\left(\mathcal{P}_{f_1}(y)\right) \end{array}$$

Since f_2 is an iso, it is clear that there is a unique $\hat{h}$ making the above diagram commute. □

We thus define Ψ on objects by

$$\Psi(\mathcal{P}) = \left(\int \mathcal{P}, \pi(\mathcal{P})\right)$$

3.2. Action of Ψ on morphisms

Let $\mathcal{P}$, $\mathcal{P}'$ be two pseudo-functors, and $\nu : \mathcal{P} \to \mathcal{P}'$ a morphism in $\mathbf{Pfct}_{\mathcal{B}}$.

Definition 3.2.1: Let

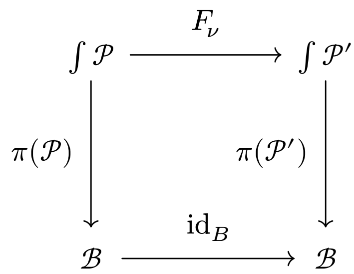
$$\begin{aligned} F_\nu : \quad \int \mathcal{P} &\longrightarrow \int \mathcal{P}' \\ (X, x) &\longmapsto (X, \nu_X(x)) \\ (f_1, f_2) &\longmapsto \left(f_1, \nu_{f_1}(b)^{-1} \circ \nu_X(f_2)\right) \end{aligned}$$

That is, for $(f_1, f_2) : (A, a) \to (B, b)$, we have

$$\begin{array}{ccc} \nu_X(a) & \xrightarrow{\quad \nu_A(f_2) \quad} & \nu_A\left(\mathcal{P}_{f_1}(b)\right) \\ & {\scriptstyle (F_\nu(f_1, f_2))_2} \searrow & \Big\downarrow {\scriptstyle \nu_{f_1}(b)^{-1}} \\ & & \mathcal{P}'_{f_1}(\nu_B(b)) \end{array}$$

Lemma 3.2.1: F_ν is a fibration morphism

Proof: We have to show that it makes the following diagram commute



and that F_ν preserves the cartisian morphisms.

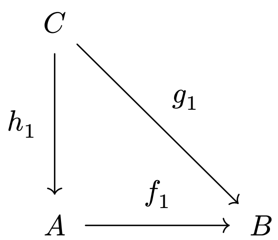
- Let's show the two functors agree:
  - on objects: let $(X, x) : \int \mathcal{P}$,

$$\begin{aligned}\pi(\mathcal{P}')(F_\nu(X, x)) &= \pi(\mathcal{P}')(X, \nu_X(x)) \\ &= X \\ &= \pi(\mathcal{P})(X, x)\end{aligned}$$

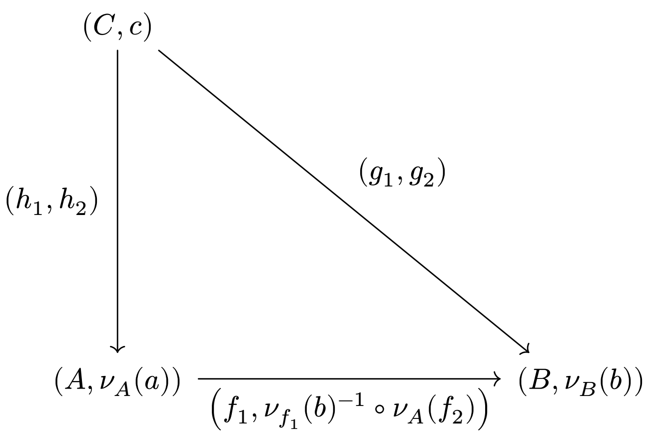
  - on morphisms: let $(f_1, f_2) : (A, a) \to (B, b)$,

$$\begin{aligned}\pi(\mathcal{P}')(F_\nu(f_1, f_2)) &= \pi(\mathcal{P}')\Big(\big(f_1, \nu_{f_1}(b)^{-1} \circ \nu_A(f_2)\big)\Big) \\ &= f_1 \\ &= \pi(\mathcal{P})(f_1, f_2)\end{aligned}$$

  Hence the diagram commutes.
- Let $(f_1, f_2) : (A, a) \to (B, b)$ be a cartesian morphism in $\int \mathcal{P}$. Let $(g_1, g_2) : (C, c) \to (B, \nu_B(b))$ be a morphism in $\int \mathcal{P}'$ and $h_1 : C \to B$ in $\mathcal{B}$ such that the following diagram commutes



  Let's show that there exists a unique $h_2 : c \to \mathcal{P}'_{h_1}(a)$ such that



  that is

$$
\begin{array}{ccc}
c & \xrightarrow{\;g_2\;} & \mathcal{P}'_{g_1}(\nu_B(b)) \\
 & & \downarrow{\scriptstyle c'_{h_1,f_1}(\nu_B(b))} \\
\Big\downarrow{\scriptstyle h_2} & & \mathcal{P}'_{h_1}\big(\mathcal{P}'_{f_1}(\nu_B(b))\big) \\
 & & \downarrow{\scriptstyle \mathcal{P}'_{h_1}\left(\nu_{f_1}(b)\right)} \\
\mathcal{P}'_{h_1}(\nu_A(a)) & \xrightarrow{\;\mathcal{P}'_{h_1}(\nu_A(f_2))\;} & \mathcal{P}'_{h_1}\Big(\nu_A\big(\mathcal{P}_{f_1}(b)\big)\Big)
\end{array}
$$

By Lemma 3.1.3, f_2 is an isomorphism, hence the commutation of the latter diagram is equivalent to that of the following, for which there clearly exists a unique h_2

$$
\begin{array}{ccc}
c & \xrightarrow{\;g_2\;} & \mathcal{P}'_{g_1}(\nu_B(b)) \\
 & & \downarrow{\scriptstyle c'_{h_1,f_1}(\nu_B(b))} \\
\Big\downarrow{\scriptstyle h_2} & & \mathcal{P}'_{h_1}\big(\mathcal{P}'_{f_1}(\nu_B(b))\big) \\
 & & \downarrow{\scriptstyle \mathcal{P}'_{h_1}\left(\nu_{f_1}(b)\right)} \\
\mathcal{P}'_{h_1}(\nu_A(a)) & \xleftarrow{\;\mathcal{P}'_{h_1}(\nu_A(f_2^{-1}))\;} & \mathcal{P}'_{h_1}\Big(\nu_A\big(\mathcal{P}_{f_1}(b)\big)\Big)
\end{array}
$$

□

4. The equivalence

4.1. $\Phi \circ \Psi$

Let $\mathcal{P} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudo-functor.

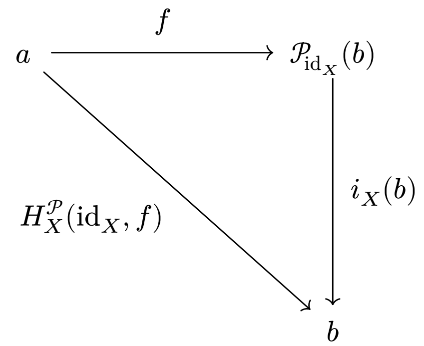
Definition 4.1.1: Consider

$$H^{\mathcal{P}} : \pi(\mathcal{P})^{-1} \longrightarrow \mathcal{P}$$

defined by, for $X : \mathcal{B}$,

$$\begin{aligned} H^{\mathcal{P}}_X :\ \pi(\mathcal{P})^{-1}_X &\longrightarrow \mathcal{P}_X \\ (X, x) &\longmapsto x \\ (\mathrm{id}_X, f) &\longmapsto i_X(b) \circ f \end{aligned}$$

Let $(\mathrm{id}_X, f) : (X, a) \to (X, b)$ in $\pi(\mathcal{P})^{-1}_X$, that is, $f : a \to \mathcal{P}_{\mathrm{id}_X}(b)$. We have



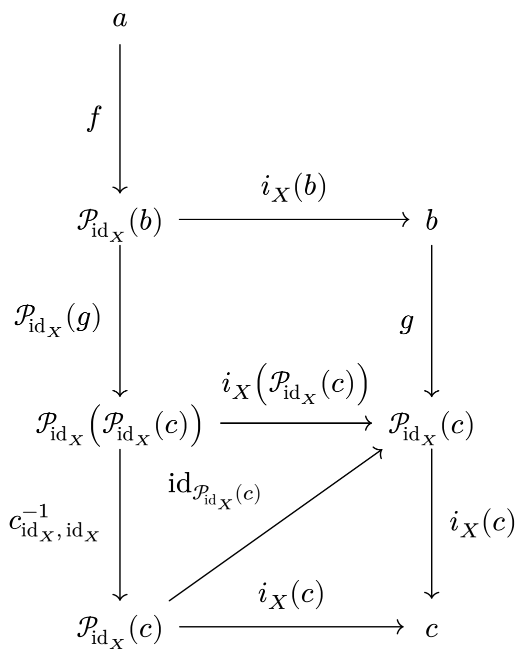
Lemma 4.1.1: For $X : \mathcal{B}$, $H^{\mathcal{P}}_X$ is a functor.

Proof: Let $(X, x) : \pi(\mathcal{P})^{-1}_X$. $\mathrm{id}_{X,x} = \left(\mathrm{id}_X, i^{-1}_X(x)\right)$, and so

$$\begin{aligned} H^{\mathcal{P}}_X\left(\mathrm{id}_{X,x}\right) &= i_X(x) \circ i^{-1}_X(x) \\ &= \mathrm{id}_x \end{aligned}$$

Furthermore, for $(X, a), (X, b), (X, c) : \pi(\mathcal{P})^{-1}_X$ and $(\mathrm{id}_X, f) : (X, a) \to (X, b)$ and $(\mathrm{id}_X, f) : (X, b) \to (X, c)$,

$$\begin{aligned} H^{\mathcal{P}}_X((\mathrm{id}_X, g) \circ (\mathrm{id}_X, f)) &= H^{\mathcal{P}}_X\left(\mathrm{id}_X, c^{-1}_{\mathrm{id}_X, \mathrm{id}_X}(c) \circ \mathcal{P}_{\mathrm{id}_X}(g) \circ f\right) \\ &= i_X(c) \circ c^{-1}_{\mathrm{id}_X, \mathrm{id}_X}(c) \circ \mathcal{P}_{\mathrm{id}_X}(g) \circ f \end{aligned}$$



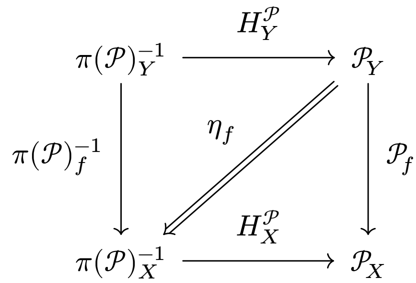
the lower right triangle commutes trivially, the triangle above commutes by a composition/identity coherence, and the square above by naturality of i_X. The outer diagram shows that

$$i_X(c) \circ c^{-1}_{\mathrm{id}_X,\mathrm{id}_X}(c) \circ \mathcal{P}_{\mathrm{id}_X}(g) \circ f = \underbrace{(i_X(c) \circ g)}_{=H^{\mathcal{P}}_X(\mathrm{id}_X, g)} \circ \underbrace{(i_X(b) \circ f)}_{=H^{\mathcal{P}}_X(\mathrm{id}_X, f)}$$

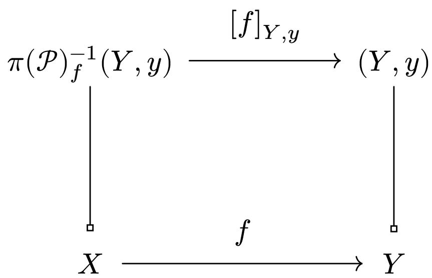
□

Lemma 4.1.2: $H^{\mathcal{P}}$ is a morphism in $\mathbf{Pfct}_{\mathcal{B}}$.

Proof: Let $f : X \to Y$ in $\mathcal{B}$, let's show that there is a natural isomorphism



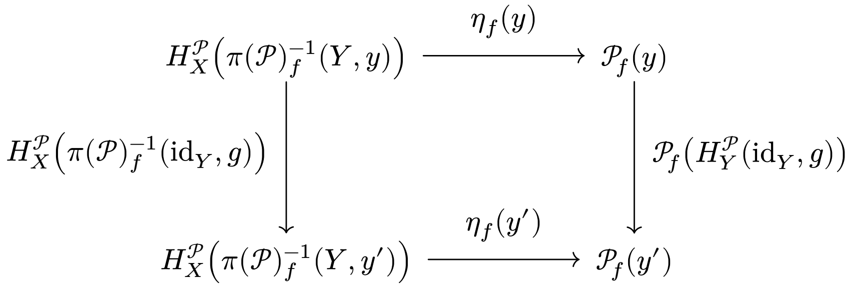
Let $(Y, y) : \pi(\mathcal{P})^{-1}_Y$, that is, $y : \mathcal{P}_Y$. We have



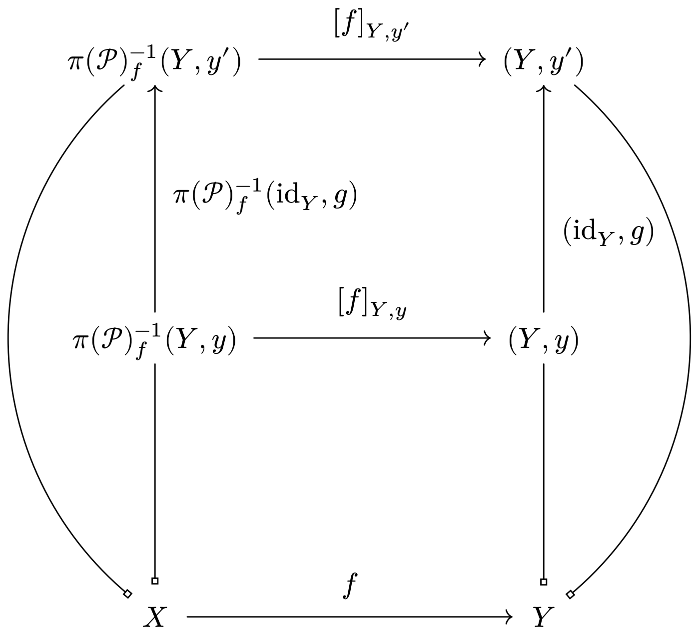
We can write $[f]_{Y,y} = \big(f, \eta_f(y)\big)$ with $\eta_f(y) : H^{\mathcal{P}}_X\Big(\pi(\mathcal{P})^{-1}_f(Y, y)\Big) \to \mathcal{P}_f(y)$.

Let's prove that η_f is natural, and that it is an isomorphism.

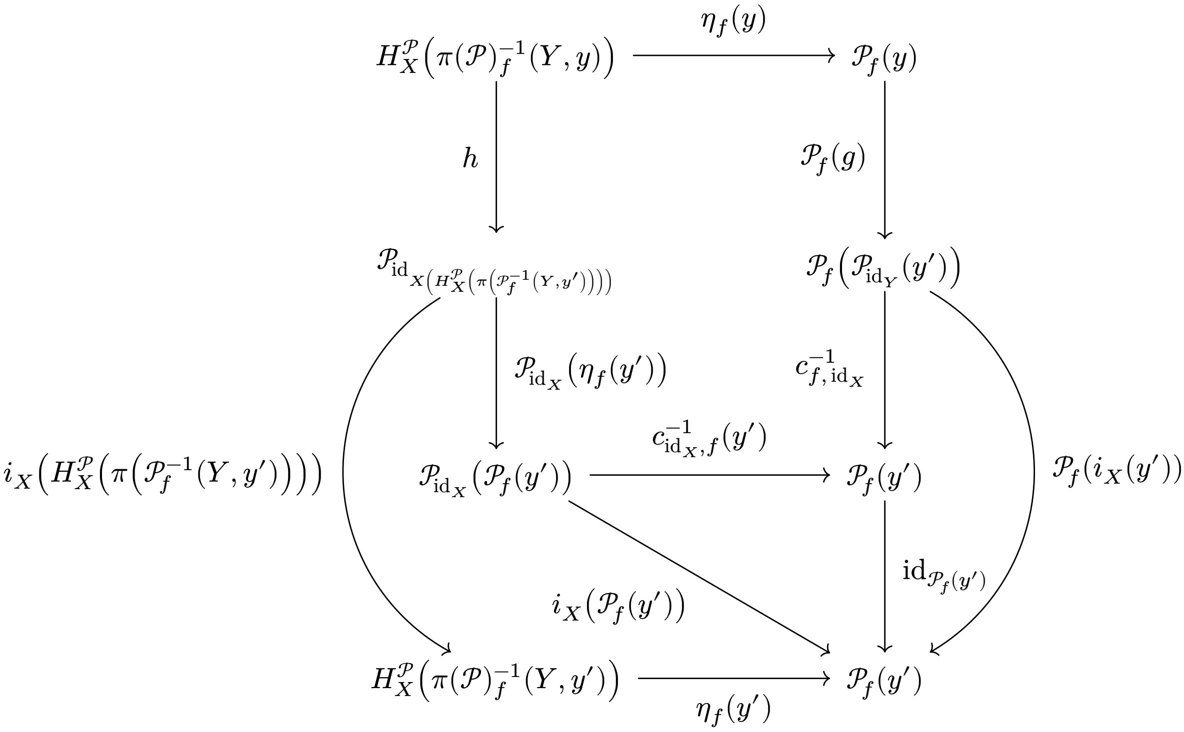
- let $y, y' : \mathcal{P}_Y$ and $g : y \to y'$. We want to show the the following diagram commutes



We have that the following diagram commutes



We can write $\pi(\mathcal{P})_f^{-1}(\mathrm{id}_Y, g) = (\mathrm{id}_Y, h)$, and so the commutation of the square implies the following commutation on the second component of the morphisms



the two triangles commute by a composition/identity coherence, while the left square is the naturality of i_X. Note that the outermost diagram is exactly the one we were looking for, showing that η_f is natural.

- $\big(f, \eta_f(y)\big) = [f]_{Y,y}$ is cartesian (by definition of $[-]_-$), hence, by Lemma 3.1.3, $\eta_f(y)$ is an isomorphism, showing that η_f is a natural isomorphism.

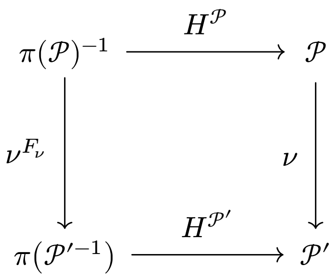
□

Lemma 4.1.3: $H^{\mathcal{P}}$ is an isomorphism.

Proof: To show that $H^{\mathcal{P}}$ is a pseudo-natural isomorphism, it is enough to show that each of its components is an isomorphism. Let $X : \mathcal{B}$. It is clear that both actions on objects and on morphisms of $H_X^{\mathcal{P}}$ are invertible. □

Lemma 4.1.4: $H^{\mathcal{P}}$ is natural in $\mathcal{P}$.

Proof: Let $\mathcal{P}, \mathcal{P}' : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ be two pseudo-functors, and $\nu : \mathcal{P} \to \mathcal{P}'$ be a pseudo-natural transformation. We have to show that



Hence, we have to show that the diagram commutes at each point $X : \mathcal{B}$

$$
\begin{array}{ccc}
\pi(\mathcal{P})_X^{-1} & \xrightarrow{H_X^{\mathcal{P}}} & \mathcal{P}_X \\
\Big\downarrow{\scriptstyle \nu_X^{F_\nu}} & & \Big\downarrow{\scriptstyle \nu_X} \\
\pi(\mathcal{P}_X'^{-1}) & \xrightarrow{H_X^{\mathcal{P}'}} & \mathcal{P}_X'
\end{array}
$$

Let's check that the functor agree on each object and morphisms:

- let $x : \mathcal{P}_X$.

$$
\begin{aligned}
H^{\mathcal{P}_X'}\left(\nu_X^{F_\nu}(X,x)\right) &= H_X^{\mathcal{P}'}(F_\nu(X,x)) \\
&= H_X^{\mathcal{P}'}(X,\nu_X(x)) \\
&= \nu_X(x) \\
&= \nu_X\left(H_X^{\mathcal{P}}(X,x)\right)
\end{aligned}
$$

- let $x, y : \mathcal{P}_X$, and $f : x \to \mathcal{P}_{\mathrm{id}_X}(y)$.

$$
\begin{aligned}
\nu_X\left(H_X^{\mathcal{P}}(\mathrm{id}_X, f)\right) &= \nu_X(i_X(y) \circ f) \\
&= \nu_X(i_X(y)) \circ \nu_X(f)
\end{aligned}
$$

  and

$$
\begin{aligned}
H_X^{\mathcal{P}'}\left(\nu_X^{F_\nu}(\mathrm{id}_X, f)\right) &= H_X^{\mathcal{P}'}(F_\nu(\mathrm{id}_X, f)) \\
&= H_X^{\mathcal{P}'}\left(\mathrm{id}_X, \nu_{\mathrm{id}_X}(y)^{-1} \circ \nu_X(f_2)\right) \\
&= i_X(\nu_X(y)) \circ \nu_{\mathrm{id}_X}(y)^{-1} \circ \nu_X(f)
\end{aligned}
$$

  We need to check that the following diagram commutes

$$
\begin{array}{ccc}
\nu_X(a) & \xrightarrow{\nu_X(f)} & \nu_X\left(\mathcal{P}_{\mathrm{id}_X}(y)\right) \\
\Big\downarrow{\scriptstyle \nu_X(f)} & \nearrow{\scriptstyle \mathrm{id}_{\nu_X\left(\mathcal{P}_{\mathrm{id}_X}(y)\right)}} & \\
\nu_X\left(\mathcal{P}_{\mathrm{id}_X}(y)\right) & & \Big\downarrow{\scriptstyle \nu_X(i_X(y))} \\
\Big\downarrow{\scriptstyle \nu_{\mathrm{id}_X}(y)^{-1}} & & \\
\mathcal{P}_{\mathrm{id}_X}(\nu_X(y)) & \xrightarrow{i_X'(\nu_X(y))} & \nu_X(y)
\end{array}
$$

  note that the lower square commutes by a coherence condition on pasting diagrams, and the upper triangle trivially commutes.

□

Lemma 4.1.5:

$$\Phi \circ \Psi \cong \mathrm{id}_{\mathbf{Pfct}_{\mathcal{B}}}$$

Proof: We have exhibited a natural isomorphism

$$H : \Phi \circ \Psi \Longrightarrow \mathrm{id}_{\mathbf{Pfct}_{\mathcal{B}}}$$

□

4.2. $\Psi \circ \Phi$

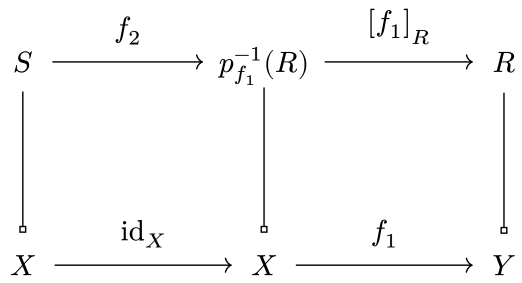
Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration.

$$\Psi \circ \Phi(p) = \pi(p^{-1}) : \int p^{-1} \to \mathcal{B}$$

Definition 4.2.1: Consider

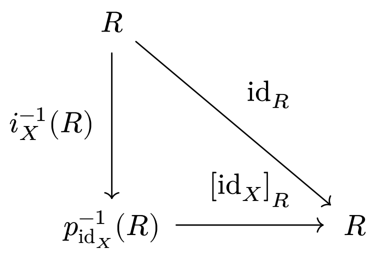
$$\begin{aligned} G_p : \int p^{-1} &\longrightarrow \mathcal{E} \\ (X, R) &\longmapsto R \\ (f_1, f_2) &\longmapsto [f_1]_R \circ f_2 \end{aligned}$$

Let $(f_1, f_2) : (X, S) \to (Y, R)$, we have $f_1 : X \to Y$ and $f_2 : S \to p_{f_1}^{-1}(R)$



Lemma 4.2.1: G_p is a functor.

Proof: Let $(X, R) : \int p^{-1}$. $\mathrm{id}_{X,R} = (\mathrm{id}_X, i_X^{-1}(R))$. We have to show that the following diagram commutes



which commutes by definition of i_X.

Furthermore, let $(X, R), (Y, S), (Z, T) : \int p^{-1}$, and

$$\begin{aligned} (f_1, f_2) &: (X, R) \longrightarrow (Y, S) \\ (g_1, g_2) &: (Y, S) \longrightarrow (Z, T) \end{aligned}$$

Let us show that $G_p((g_1, g_2) \circ (f_1, f_2)) = G_p(g_1, g_2) \circ G_p(f_1, f_2)$, that is, that the following diagram commutes

$$\begin{array}{ccc}
R & \xrightarrow{\quad f_2 \quad} & p_{f_1}^{-1}(S) \\
{\scriptstyle f_2}\Big\downarrow & & \Big\downarrow{\scriptstyle [f_1]_S} \\
p_{f_1}^{-1}(S) & & S \\
{\scriptstyle p_{f_1}^{-1}(g_2)}\Big\downarrow & & \Big\downarrow{\scriptstyle g_2} \\
p_{f_1}^{-1}\left(p_{g_1}^{-1}(T)\right) & & p_{g_1}^{-1}(T) \\
{\scriptstyle [f_1, g_1]_R^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle [g_1]_T} \\
p_{g_1 \circ f_1}^{-1}(T) & \xrightarrow{\quad [g \circ f]_T \quad} & T
\end{array}$$

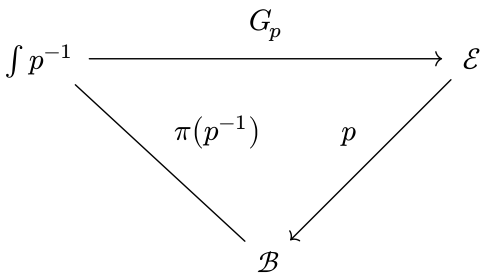
We indeed have the following diagram commutes

$$\begin{array}{ccc}
R & \xrightarrow{\quad f_2 \quad} & p_{f_1}^{-1}(S) \\
{\scriptstyle f_2}\Big\downarrow & & \Big\downarrow{\scriptstyle [f_1]_S} \\
p_{f_1}^{-1}(S) & \xrightarrow{\quad [f_1]_S \quad} & S \\
{\scriptstyle p_{f_1}^{-1}(g_2)}\Big\downarrow & & \Big\downarrow{\scriptstyle g_2} \\
p_{f_1}^{-1}\left(p_{g_1}^{-1}(T)\right) & \xrightarrow{\quad [f_1]_{p_{g_1}^{-1}(T)} \quad} & p_{g_1}^{-1}(T) \\
{\scriptstyle [f_1, g_1]_R^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle [g_1]_T} \\
p_{g_1 \circ f_1}^{-1}(T) & \xrightarrow{\quad [g \circ f]_T \quad} & T
\end{array}$$

as the lower square commutes by definition of $[f_1, g_1]_R$, the middle one by definition of $p_{f_1}^{-1}(g_2)$, and the top one commutes trivially. □

Lemma 4.2.2: G_p is a fibration morphism.

Proof: There are two things to check: the commutation with the fibrations, and the preservation of cartesian morphisms. Let's proceed in order.
1.



Let's check that the two functors agree on objects and morphisms.

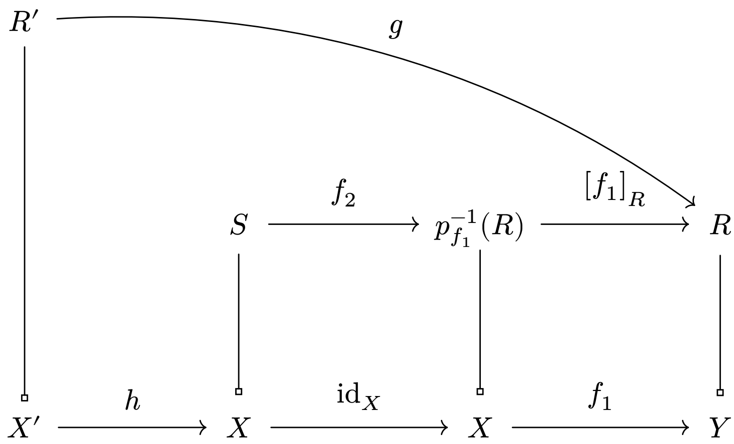
- let $(X, x) : \int p^{-1}$, ie $X = p(x)$

$$\begin{aligned} p\big(G_p(X, x)\big) &= p(x) \\ &= X \\ &= \pi(p^{-1})(X, x) \end{aligned}$$

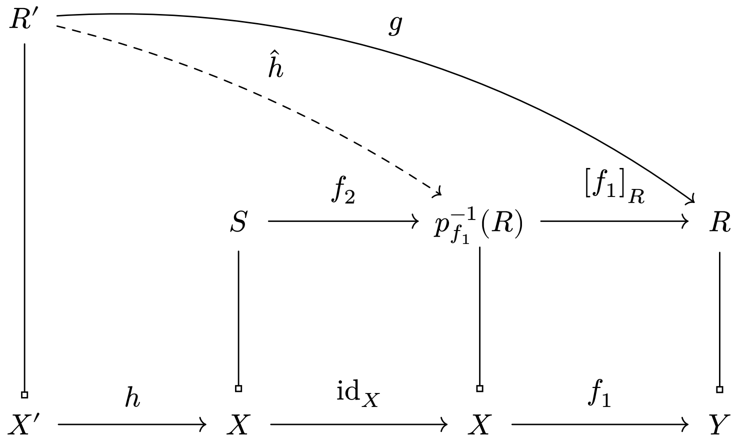
- let $(X, x), (Y, y) : \int p^{-1}$, and $(f_1, f_2) : (X, x) \to (Y, y)$. We have

$$\begin{aligned} p\big(G_p(f_1, f_2)\big) &= p\Big([f_1]_y \circ f_2\Big) \\ &= p\Big([f_1]_y\Big) \circ p(f_2) \\ &= f_1 \circ \mathrm{id}_X \\ &= f_1 \\ &= \pi(p^1)(f_1, f_2) \end{aligned}$$

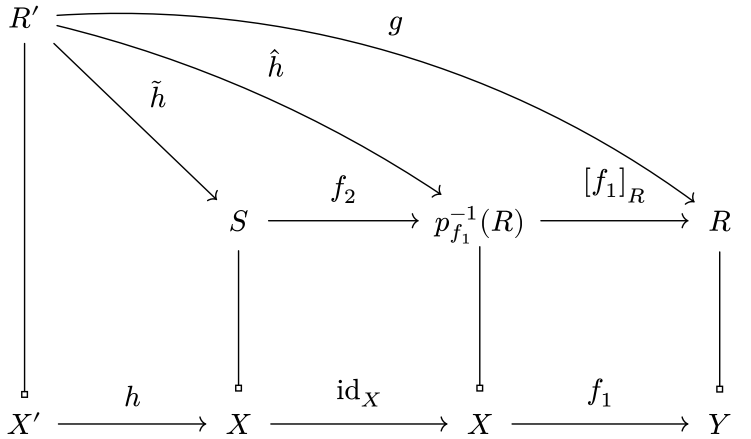
2. Let $(f_1, f_2) : (X, S) \to (Y, R)$ be a cartesian morphism. By Lemma 3.1.3, f_2 is an isomorphism. Let $h : X' \to X$ and $g : R' \to R$ such that the following diagram commutes



By cartesianity of $[f_1]_R$, there exists a unique $\hat{h} : R' \to p_{f_1}^{-1}(R)$ such that the following diagram commutes



Hence, $f_2^{-1} \circ \hat{h}$ satisfies the wanted property. Furthermore, for any $\tilde{h} : R' \to S$ that makes the following diagram commute



note that $f_2 \circ \tilde{h}$ satisfies the same universal property as $\hat{h}$, hence $f_2 \circ \tilde{h} = \hat{h}$, and thus

$$\tilde{h} = f_2^{-1} \circ \hat{h}$$

which shows the unicity.

□

Lemma 4.2.3: G_p is an isomorphism.

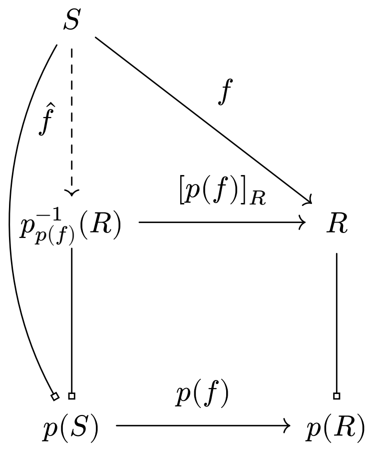
Proof: Let us exhibit an inverse morphism

$$K_p : \mathcal{E} \longrightarrow \int p^{-1}$$

- if $R : \mathcal{E}$, we define

$$K_p(R) = (p(R), R)$$

- if $S, R : \mathcal{E}$ and $f : S \longrightarrow R$ is a morphism in $\mathcal{E}$, by cartesianity of $[p(f)]_R$, there exists a unique $\hat{f} : S \to p_{p(f)}^{-1}(R)$ such that the following diagram commutes

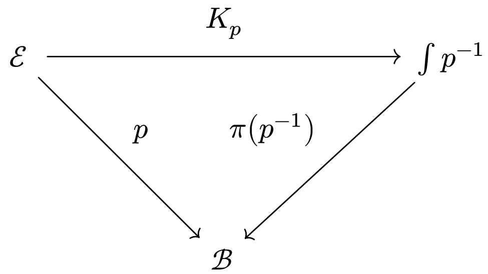


Let

$$K_p = \big(p(f), \hat{f}\big)$$

Let us show that K_p is the inverse of G_p (which will entail that it is a functor), and that it is a fibration morphism.

1. It is clear that K_p and G_p are each other's inverse on objects. Let (f_1, f_2) be a morphism in $\int p^{-1}$. We have that $p\big([f_1]_R \circ f_2\big) = p\big([f_1]_R\big) \circ p(f_2) = f_1 \circ \mathrm{id} = f_1$. Furthermore, f_2 is precisely the cartesian lifting of the identity by $[f_1]_R$, so we have $K_p\big(G_p(f_1, f_2)\big) = (f_1, f_2)$. Conversely, let $f : S \to R$ be a morphism in $\mathcal{E}$. By definition of $\hat{f}$, we have $f = [p(f)] \circ \hat{f}$, so $G_p\big(K_p(f)\big) = f$.
2. We have to check that the following diagram commutes



Let's check that the two functors agree on objects and morphisms.

- Let $R : \mathcal{E}$

$$\begin{aligned}\pi(p^1)\big(K_p(R)\big) &= \pi(p^{-1})(p(R), R)\\ &= p(R)\end{aligned}$$

- Let $S, R : \mathcal{E}$ and $f : S \to R$ a morphism in $\mathcal{E}$

$$\begin{aligned}\pi(p^{-1})\big(K_p(f)\big) &= \pi(p^{-1})\big(p(f), \hat{f}\big)\\ &= p(f)\end{aligned}$$

Furthermore, we have to check that K_p preserves cartesian morphisms. Let f be cartesian. $\hat{f}$ is (the canonical) isomorphism between the domains living in the same fiber, of two cartesian morphisms. In particular, it is an isomorphism, hence $\big(p(f), \hat{f}\big)$ is cartesian by Lemma 3.1.4.

□

Lemma 4.2.4: G_p is natural in p.

Proof: Let $p : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{F} \to \mathcal{B}$ be two fibrations, and $F : p \to q$ be a morphism of fibrations. Let's check that the following diagram commutes

$$\begin{array}{ccc} \int p^{-1} & \xrightarrow{G_p} & \mathcal{E} \\ {\scriptstyle F_{\nu^F}} \downarrow & & \downarrow {\scriptstyle F} \\ \int q^{-1} & \xrightarrow{G_q} & \mathcal{F} \end{array}$$

Let's check that the two functors agree on objects and morphisms. Let $(X, R) : \int p^{-1}$.

$$\begin{aligned} G_q(F_{\nu^F}(X, R)) &= G_q(X, \nu^F(R)) \\ &= \nu^F_X(R) \\ &= F(R) \\ &= F\big(G_p(X, R)\big) \end{aligned}$$

Let $(X, R), (Y, S) : \int p^{-1}$ and $(f_1, f_2) : (X, R) \to (Y, S)$ be a morphism in $\int p^{-1}$.

$$\begin{aligned} G_q(F_{\nu^F}(f_1, f_2)) &= G_q\Big(f_1, \nu_{f_1^F}(S)^{-1} \circ \nu^F_X(f_2)\Big) \\ &= [f_1]_{F(R)} \circ \nu_{f_1^F}(S)^{-1} \circ \nu^F_X(f_2) \\ &= [f_1]_{F(R)} \circ \nu_{f_1^F}(S)^{-1} \circ F(f_2) \\ &= F\Big([f_1]_R\Big) \circ F(f_2) \\ &= F\Big([f_1]_R \circ f_2\Big) \\ &= F\big(G_p(f_1, f_2)\big) \end{aligned}$$

□

Lemma 4.2.5:

$$\Psi \circ \Phi \cong \mathrm{id}_{\mathbf{Fib}_{\mathcal{B}}}$$

Proof: We have exhibited the natural isomorphism

$$G : \Psi \circ \Phi \Longrightarrow \mathrm{id}_{\mathbf{Fib}_{\mathcal{B}}}$$

□

This concludes the proof of the main theorem.