Grothendieck fibrations

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Fibration

Refinement system

Assume that we have two categories $\mathcal{B}$ and $\mathcal{E}$, and a functor $p : \mathcal{E} \to \mathcal{B}$.

Definition

For $R : \mathcal{E}$ and $X : \mathcal{B}$, we say that R *refines* X, or $R \sqsubset X$ if

$$X = p(R)$$

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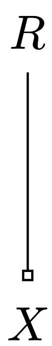

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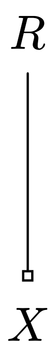

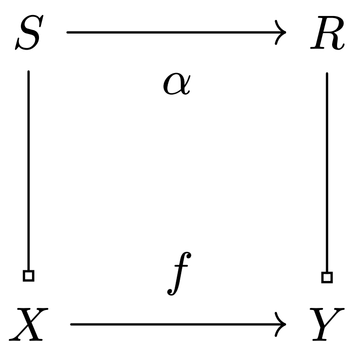

Cartesian morphism

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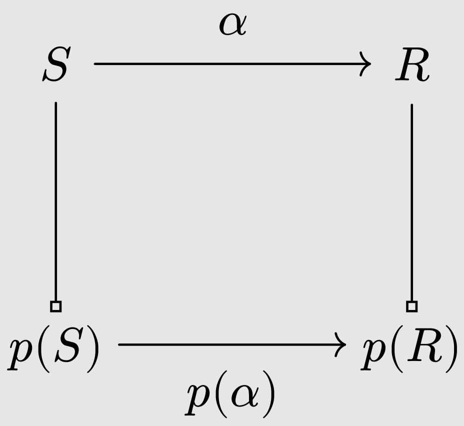
A morphism $\alpha : S \to R$ in $\mathcal{E}$ is *cartesian*



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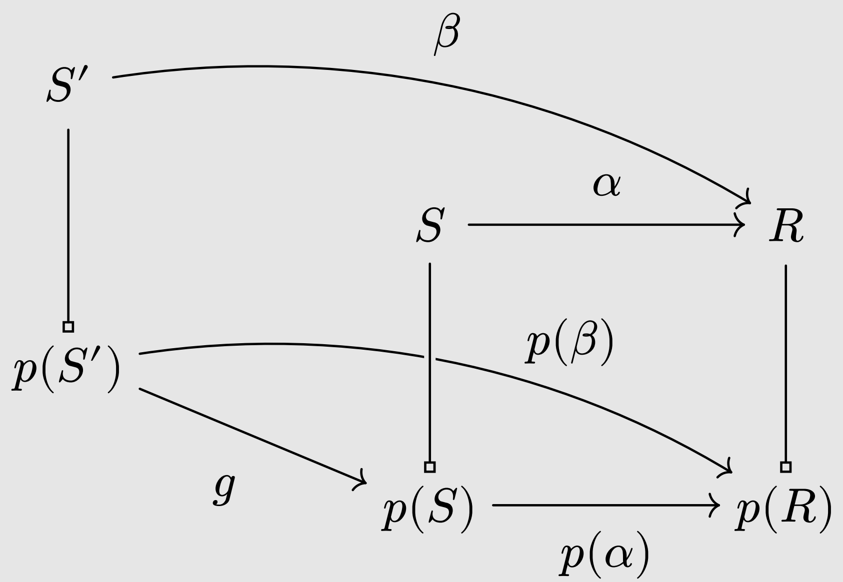
A morphism $\alpha : S \to R$ in $\mathcal{E}$ is *cartesian* when, for any morphism $\beta : S' \to R$ in $\mathcal{E}$ and $g : p(S') \to p(S)$ such that

$$p(\beta) = p(\alpha) \circ g$$



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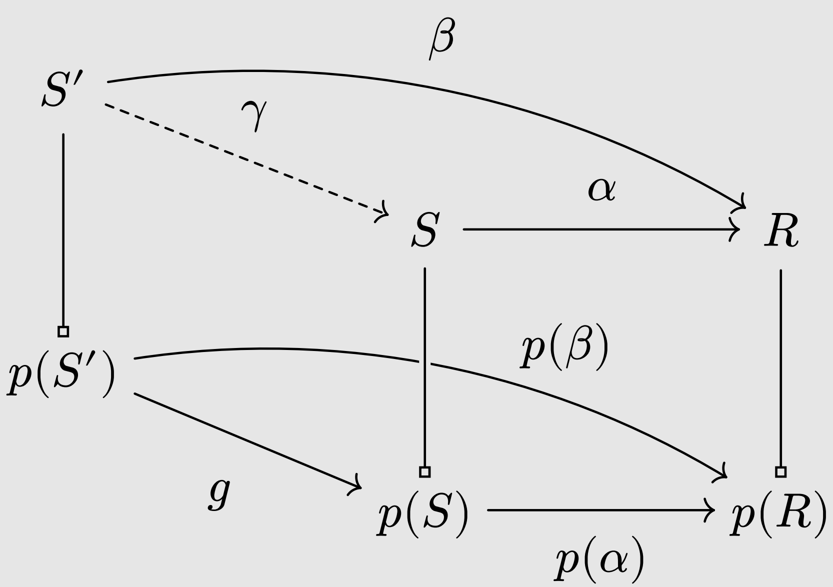
there exists a unique lifting $\gamma : S' \to S$ of g (ie $p(\gamma) = g$) such that

$$\beta = \alpha \circ \gamma$$



Fibration

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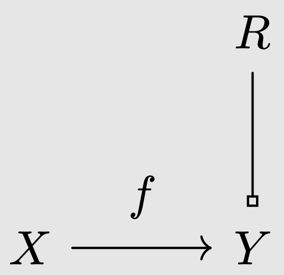
Definition

A functor $p : \mathcal{E} \to \mathcal{B}$ is a *fibration* if, for any $f : X \to Y$ in $\mathcal{B}$, and $R \sqsubset Y$



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$$\begin{array}{ccc} S & \xrightarrow{\alpha} & R \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Fibration

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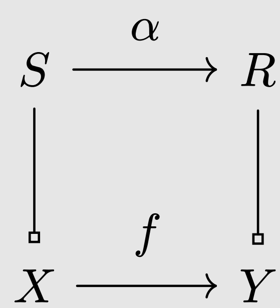
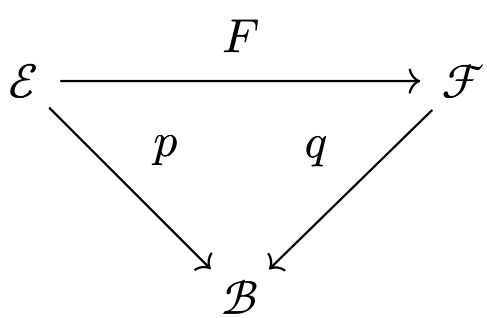


Given two fibrations $p : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{F} \to \mathcal{B}$, a *fibration morphism* $F : \mathcal{E} \to \mathcal{F}$ is a functor

- that preserves cartesian morphisms
- such that the following diagram commutes



Let $\mathbf{Fib}_{\mathcal{B}}$ be the category of fibration and fibration morphisms over $\mathcal{B}$.

Example: the codomain fibration

Consider the category $\mathcal{B}^{\to}$ whose objects are morphisms $f : X \to Y$ in $\mathcal{B}$. For $f : X \to Y$ and $g : A \to B$ two objects of $\mathcal{B}^{\to}$, a morphism $(\varphi_1, \varphi_2) : f \to g$ is a pair of morphisms

$$\varphi_1 : X \to A$$
$$\varphi_2 : Y \to B$$

such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\varphi_1} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\varphi_2} & B \end{array}$$

with obvious identities and compositions.

Definition

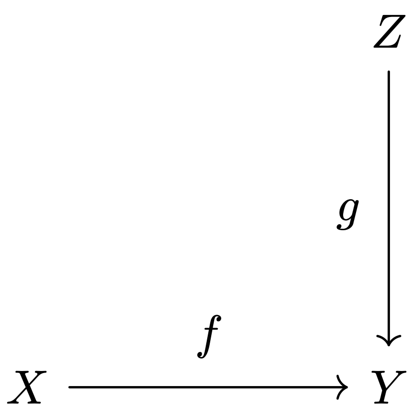
The codomain functor **cod** : $\mathcal{B}^{\to} \to \mathcal{B}$ maps an object $f : X \to Y$ to Y, and a morphism (φ_1, φ_2) to φ_2.

Example: the codomain fibration

Theorem

If $\mathcal{B}$ has pullbacks, then **cod** is a fibration.

Indeed, consider $X, Y : \mathcal{B}$ and $f : X \to Y$, as well as $g : Z \to Y$ $(g \sqsubset Y)$. We have



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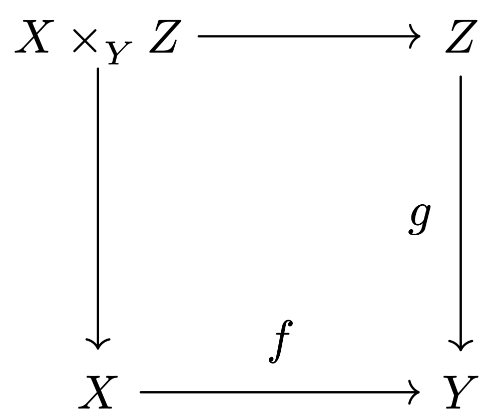

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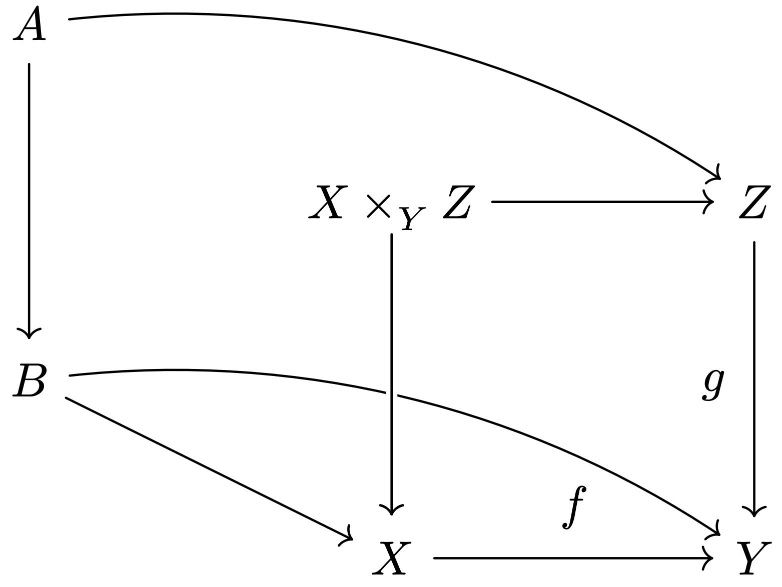

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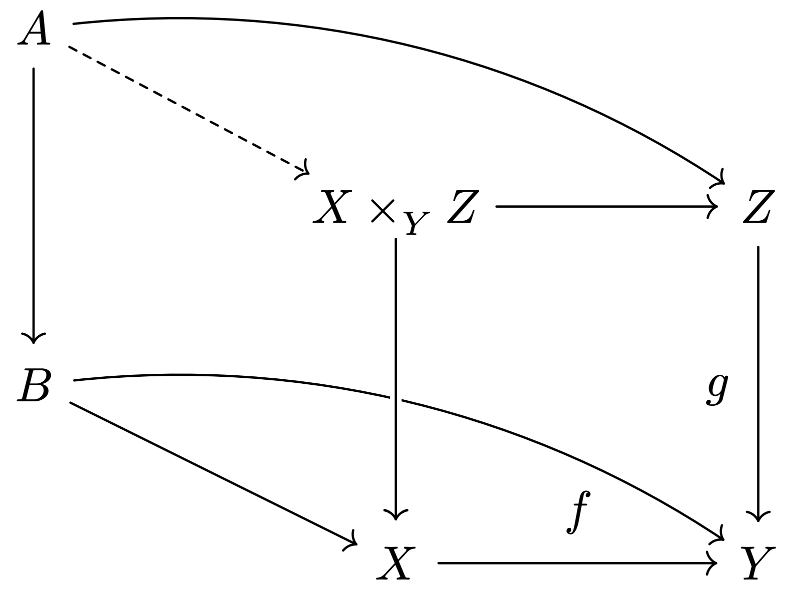


Fibered category

Pseudofunctor

Definition

We note $\mathbf{Pfct}_{\mathcal{B}}$ the category of contravariant pseudofunctor from $\mathcal{B}$ to $\mathbf{Cat}$.

An object $\mathcal{P} : \mathbf{Pfct}_{\mathcal{B}}$ is the data of:

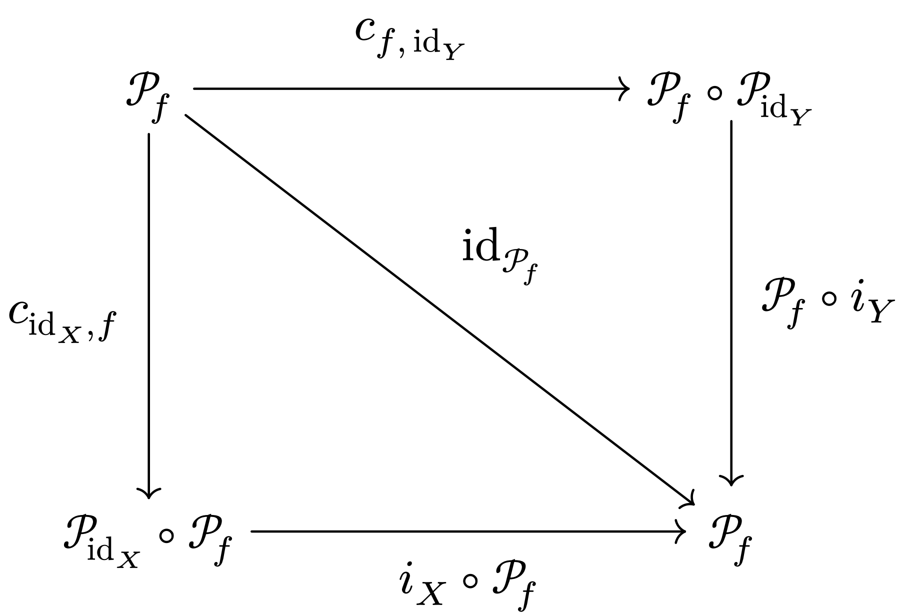
- for $X : \mathcal{B}$, a category $\mathcal{P}_X : \mathbf{Cat}$;
- for $f : X \to Y$ a morphism in $\mathcal{B}$, a functor $\mathcal{P}_f : \mathcal{P}_Y \to \mathcal{P}_X$;
- for $X : \mathcal{B}$, a natural isomorphism $i_X : \mathcal{P}_{\mathrm{id}_X} \Rightarrow \mathrm{id}_{\mathcal{P}_X}$, called the *pseudo identity* of $\mathcal{P}$ at X;
- for f, g two morphism in $\mathcal{B}$, a natural isomorphism $c_{f,g} : \mathcal{P}_{g \circ f} \Longrightarrow \mathcal{P}_f \circ \mathcal{P}_g$, called the *pseudo composition* of $\mathcal{P}$ at (f, g).

Satisfying additionally two coherence conditions.

Pseudofunctor

Identity/composition coherence

For $X, Y : \mathcal{B}$ and $f : X \to Y$, we have



Pseudofunctor

Composition/composition coherence

For $W, X, Y, Z : \mathcal{B}$, and

$$
\begin{aligned}
f &: W \longrightarrow X \\
g &: X \longrightarrow Y \\
h &: Y \longrightarrow Z
\end{aligned}
$$

the following diagram commutes

$$
\begin{CD}
\mathcal{P}_{h \circ g \circ f} @>{c_{f, h \circ g}}>> \mathcal{P}_f \circ \mathcal{P}_{h \circ g} \\
@V{c_{g \circ f, h}}VV @VV{\mathcal{P}_f \circ c_{g,h}}V \\
\mathcal{P}_{g \circ f} \circ \mathcal{P}_h @>{c_{f,g} \circ \mathcal{P}_h}>> \mathcal{P}_f \circ \mathcal{P}_g \circ \mathcal{P}_h
\end{CD}
$$

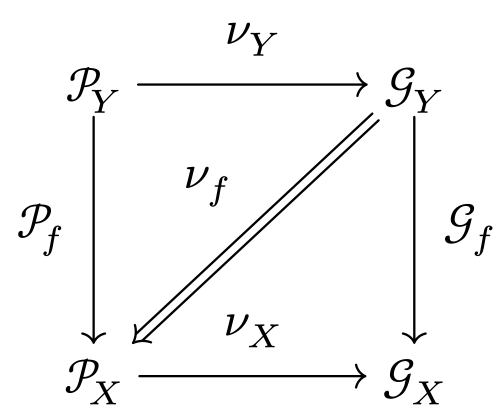
Pseudofunctor morphism

Definition

Let $\mathcal{F}$, $\mathcal{G}$: $\mathbf{Pfct}_{\mathcal{B}}$ be two pseudofunctors. A morphism $\nu : \mathcal{F} \to \mathcal{G}$ is a pseudonatural transformation between $\mathcal{F}$ and $\mathcal{G}$.

That is, a morphism ν is the data:

- for $X : \mathcal{B}$, a morphism $\nu_X : \mathcal{F}_X \to \mathcal{G}_X$;
- for each morphism $f : X \to Y$, a natural transformation

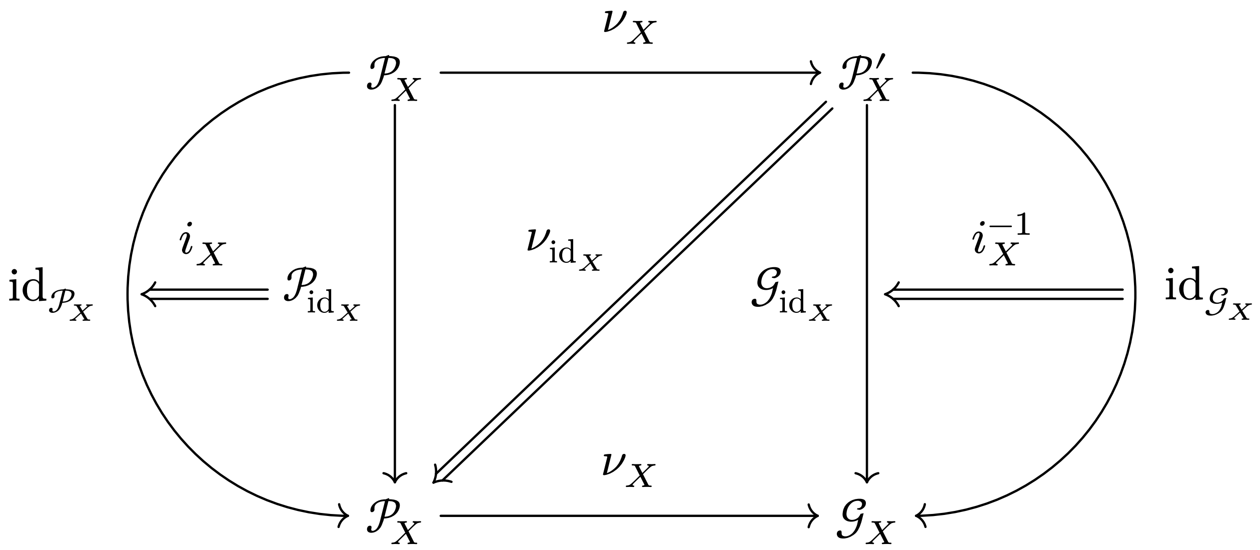


called the *pseudo naturality* of ν at f.

ν satisfies additionally two coherence conditions.

Pseudofunctor morphism

For $X : \mathcal{B}$, the following pasting diagram is ν_X

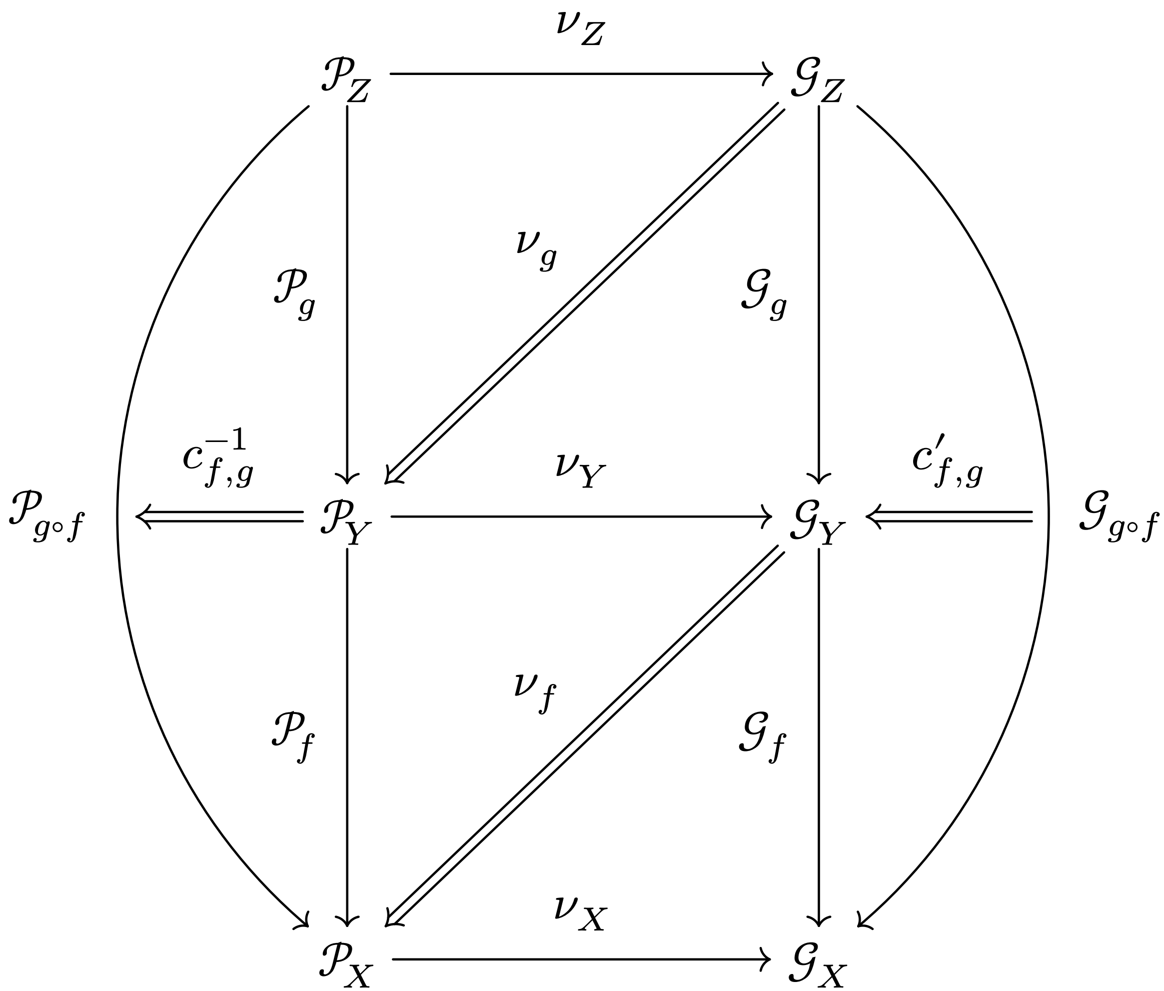


that is,

$$(\nu_X \circ i_X) \circ \nu_{\mathrm{id}_X} \circ (i_X'^{-1} \circ \nu_X) = \mathrm{id}_{\nu_X}$$

Pseudofunctor morphism

For $X, Y, Z : \mathcal{B}$, $f : X \to Y$ and $g : Y \to Z$, the following pasting is $\nu_{g \circ f}$



that is,

$$\nu_{g \circ f} = \left(\nu_X \circ c_{f,g}^{-1}\right) \circ \left(\nu_f \circ \mathcal{P}_g\right) \circ \left(\mathcal{G}_f \circ \nu_g\right) \circ \left(c'_{f,g} \circ \nu_Z\right)$$

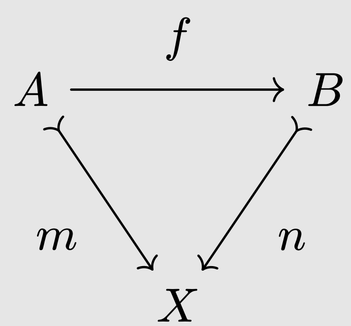
Example: the subobject pseudofunctor

For this example, assume that $\mathcal{B}$ has pullbacks.

Let $X : \mathcal{B}$ be an object.

Definition

The subobjects of X is the category $\mathbf{Sub}_X$ whose objects are monos into X, up to isomorphism. Given $m, m' : \mathbf{Sub}_X$, a morphism $f : m \to m'$ is a morphism in $\mathcal{B}$ making the following diagram commute

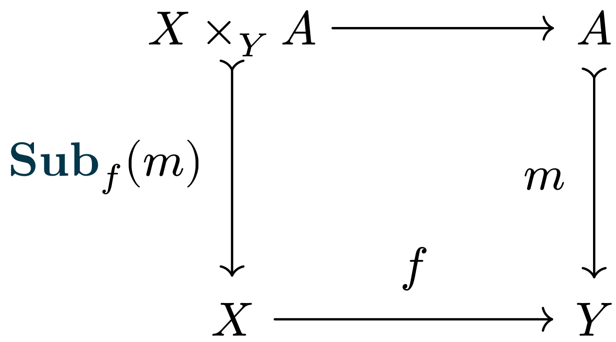


Morphisms in $\mathbf{Sub}_X$ are all monos in $\mathcal{B}$, and between two objects there is at most one morphism, making $\mathbf{Sub}_X$ a poset category. Hence, we write $m \leq n$ if there is a morphism $m \to n$.

Example: the subobject pseudofunctor

Let $X, Y : \mathcal{B}$ and $f : X \to Y$. If $m : \mathbf{Sub}_Y$, say $m : A \rightarrowtail Y$.

One can consider $\mathbf{Sub}_f(m)$ defined by taking the following pullback



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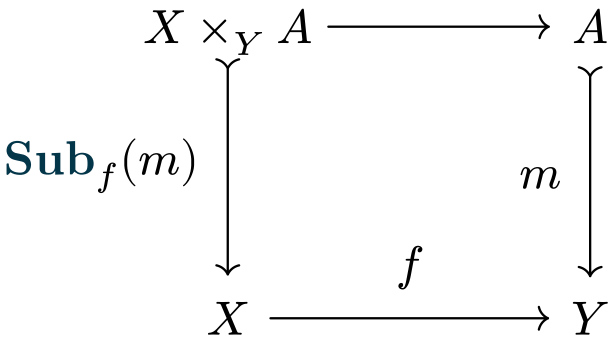
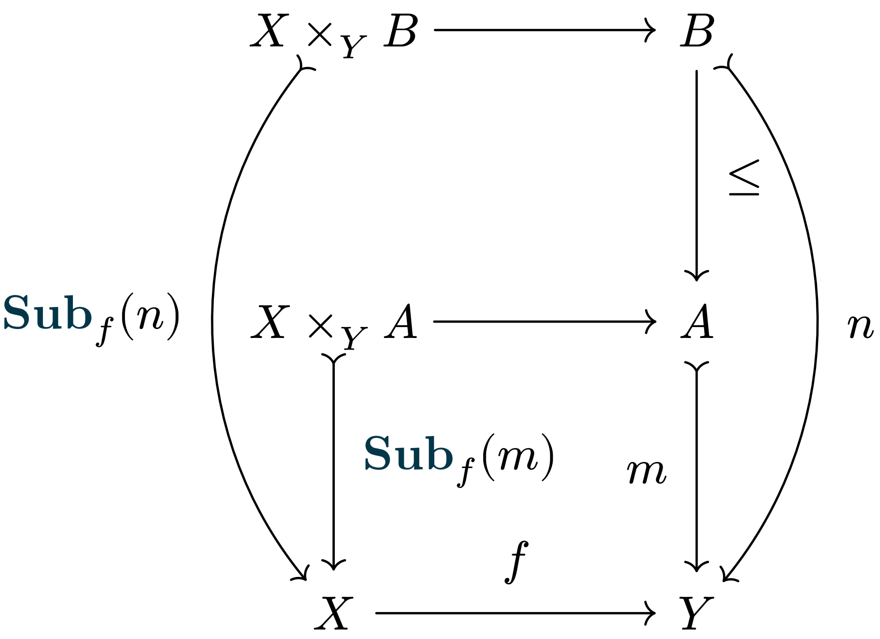


Suppose $n \leq m$:



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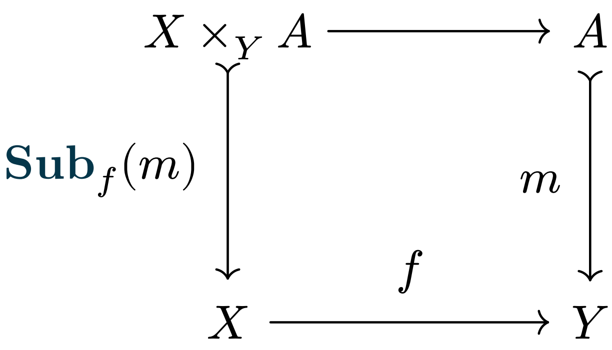

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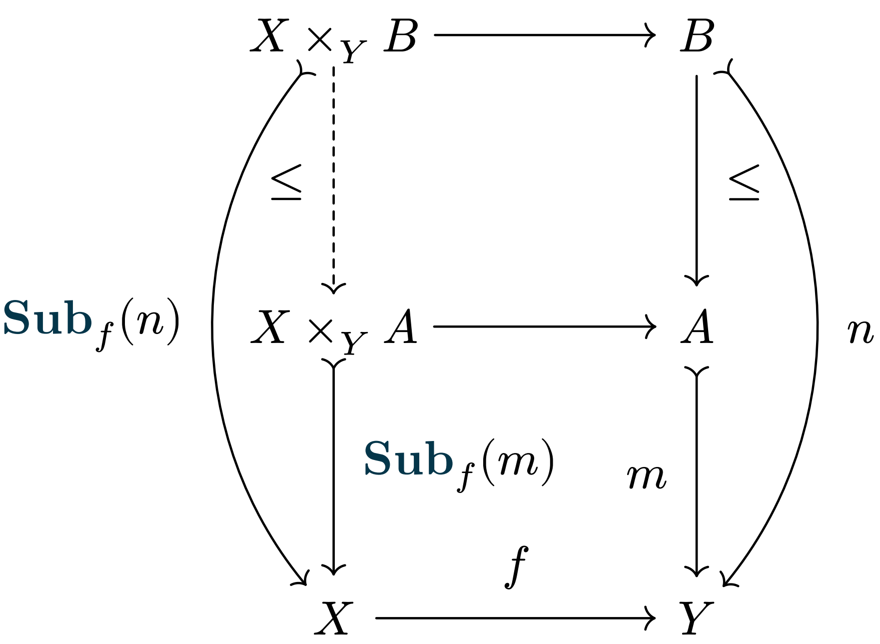

by pullback, there exists a map $X \times_Y B \to X \times_Y A$ making the left triangle commute.

Example: the subobject pseudofunctor

We have shown that $\mathbf{Sub}_X$ and $\mathbf{Sub}_Y$ are categories, and, for $f : X \to Y$,

$$\mathbf{Sub}_f : \mathbf{Sub}_Y \to \mathbf{Sub}_X$$

is a functor. In fact, this operation is itself functorial in f, proving the

Theorem

Subobjects

$$\mathbf{Sub} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$$

form a presheaf into categories.

The proof goes by pasting pullback diagrams together to form pullback diagrams.

Equivalence

Main theorem

For a category $\mathcal{B}$, we have the following equivalence

Theorem

$$\mathbf{Fib}_{\mathcal{B}} \cong \mathbf{Pfct}_{\mathcal{B}}$$

This equivalence states that a fibration is exactly the collection of its fibers.

One direction of this equivalence is known as the Grothendieck construction, which takes a collection of fibers above a category $\mathcal{B}$ and constructs a fibration from its total category, on $\mathcal{B}$, whose fibers are exactly those we started with.