

Notes on (Co)end Calculus

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1. Dinaturality and co/ends

Exercise 1.1

Prove equations (1.5) and (1.6), that is, for a morphism $f : B \rightarrow B'$ and $u : A \rightarrow X^{B'}$,

$$\epsilon_{X,B'} \circ (u \times B') \circ (A \times f) = \epsilon_{X,B} \circ (X^f \circ u) \times B \quad (1.5)$$

$$\epsilon_{X,B'} \circ (X^{B'} \times f) \circ (u \times B) = \epsilon_{X,B} \circ (X^f \times B) \circ (u \times B) \quad (1.6)$$

Proof. Naturality in B of the isomorphism $\mathbf{Set}(A, X^B) \cong \mathbf{Set}(A \times B, X)$ states that the following diagram commutes

$$\begin{array}{ccc} \mathbf{Set}(A, X^{B'}) & \xrightarrow{\text{ev}_{A,X,B'}} & \mathbf{Set}(A \times B', X) \\ \downarrow \mathbf{Set}(A, X^f) & & \downarrow \mathbf{Set}(A \times f, X) \\ \mathbf{Set}(A, X^B) & \xrightarrow{\text{ev}_{A,X,B}} & \mathbf{Set}(A \times B, X) \end{array}$$

hence

$$(\mathbf{Set}(A \times f, X) \circ \text{ev}_{A,X,B'})(u) = (\text{ev}_{A,X,B} \circ \mathbf{Set}(A, X^f))(u)$$

where $\text{ev}_{A,X,B'}(u) = \epsilon_{X,B'} \circ (u \times B')$. Thus

$$\begin{array}{ccc} A \times B & \xrightarrow{A \times f} & A \times B' \\ \downarrow & & \downarrow u \times B' \\ (X^f \circ u) \times B & & X^{B'} \times B' \\ \downarrow & & \downarrow \epsilon_{X,B'} \\ X^B \times B & \xrightarrow{\epsilon_{X,B}} & X \end{array}$$

which is exactly equation (1.5).

(1.6) follows immediately from (1.5) as $(X^f \circ u) \times B = (X^f \times B) \circ (u \times B)$ and $(u \times B') \circ (A \times f) = (X^{B'} \times f) \circ (u \times B)$, by functoriality of $- \times -$. \square

Exercise 1.2

1. For \mathcal{C} and \mathcal{D} two categories, and $P, Q, R : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ functors, show with an example that dinatural transformations $\alpha : P \rightrightarrows Q$ and $\beta : Q \rightrightarrows R$ cannot be composed in general.
2. Show, however, that dinatural transformations and natural transformations can be composed to form dinatural transformations.

Proof.

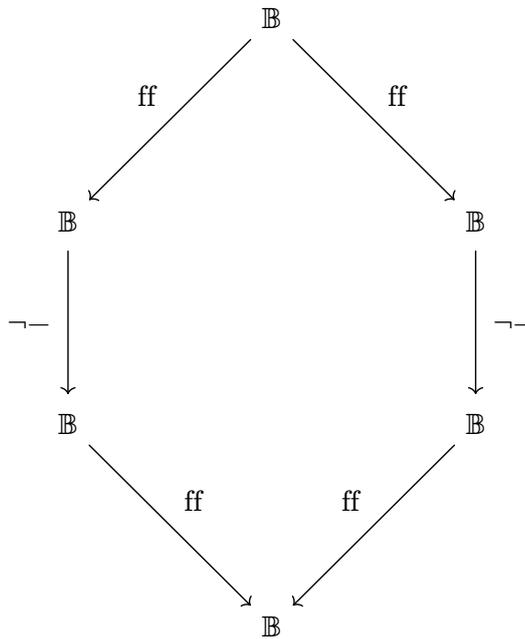
1. Consider $\mathcal{C} = \mathbf{2}$, and $\mathcal{D} = \{\mathbb{B}\}$ be a full subcategory of **Set**. Let

$$\begin{aligned} P : \mathbf{2}^{\text{op}} \times \mathbf{2} &\longrightarrow \{\mathbb{B}\} \\ (x, y) &\longmapsto \mathbb{B} \\ (f, g) &\longmapsto \lambda x. \text{ff} \end{aligned}$$

and $Q = P$. Let

$$\begin{aligned} \alpha : P &\rightrightarrows Q \\ x &\longmapsto \neg - \end{aligned}$$

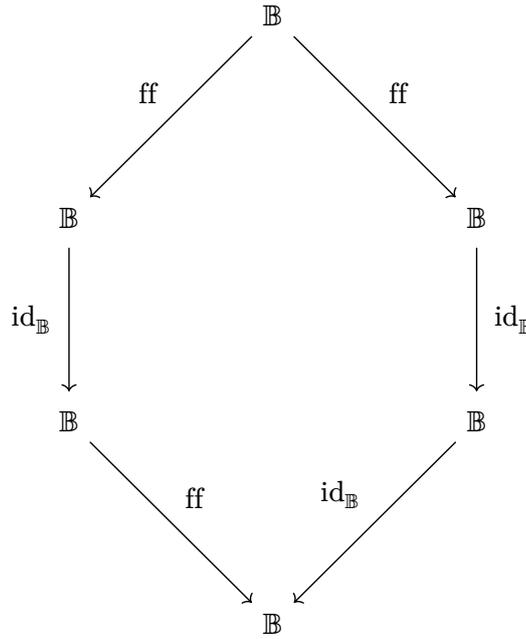
This is a dinatural transformation. Indeed consider the following dinaturality condition diagram at $0 < 1$



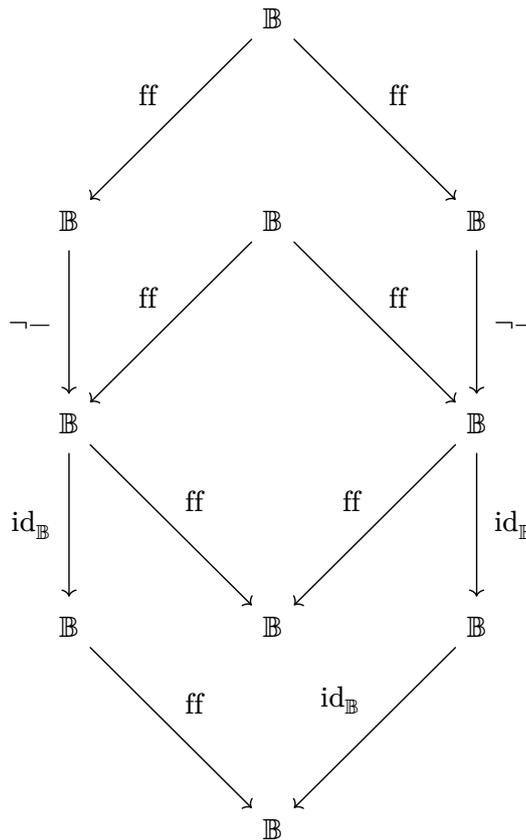
It clearly commutes. Now, consider

$$\begin{aligned} R : \mathbf{2}^{\text{op}} \times \mathbf{2} &\longrightarrow \{\mathbb{B}\} \\ (x, y) &\longmapsto \mathbb{B} \\ (0 < 1, \text{id}) &\longmapsto \text{id}_{\mathbb{B}} \\ (\text{id}, 0 < 1) &\longmapsto \lambda x. \text{ff} \end{aligned}$$

and have $\beta : Q \rightrightarrows R$ be identity pointwise. β is a dinatural transformation. Indeed consider the following dinaturality condition diagram at $0 < 1$



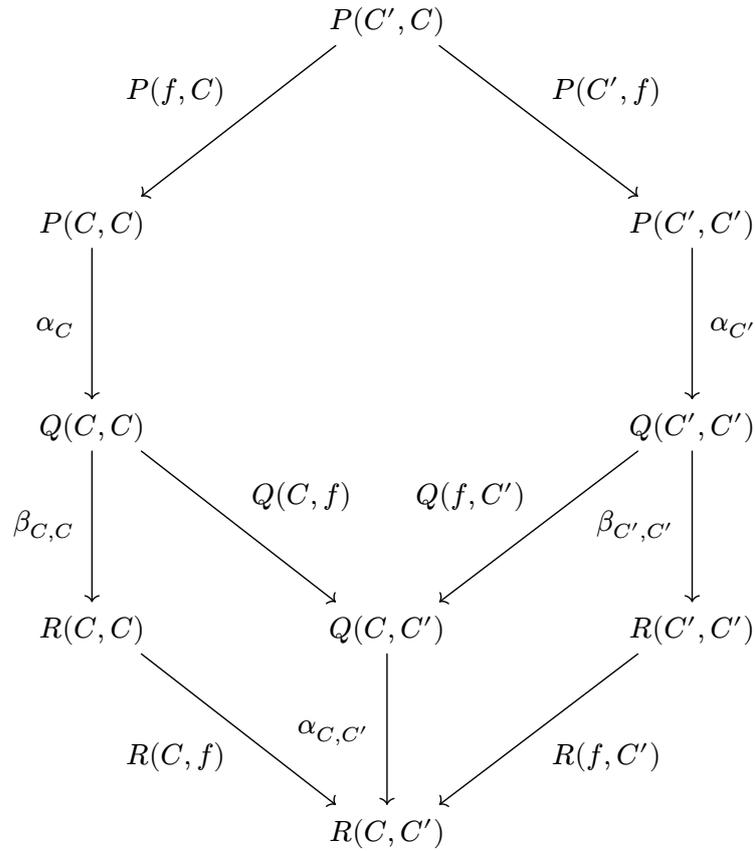
Again, this clearly commutes. Yet, the composition $\beta \circ \alpha$ is not a dinatural transformation, as can be witnessed by looking the dinaturality condition diagram at $0 < 1$



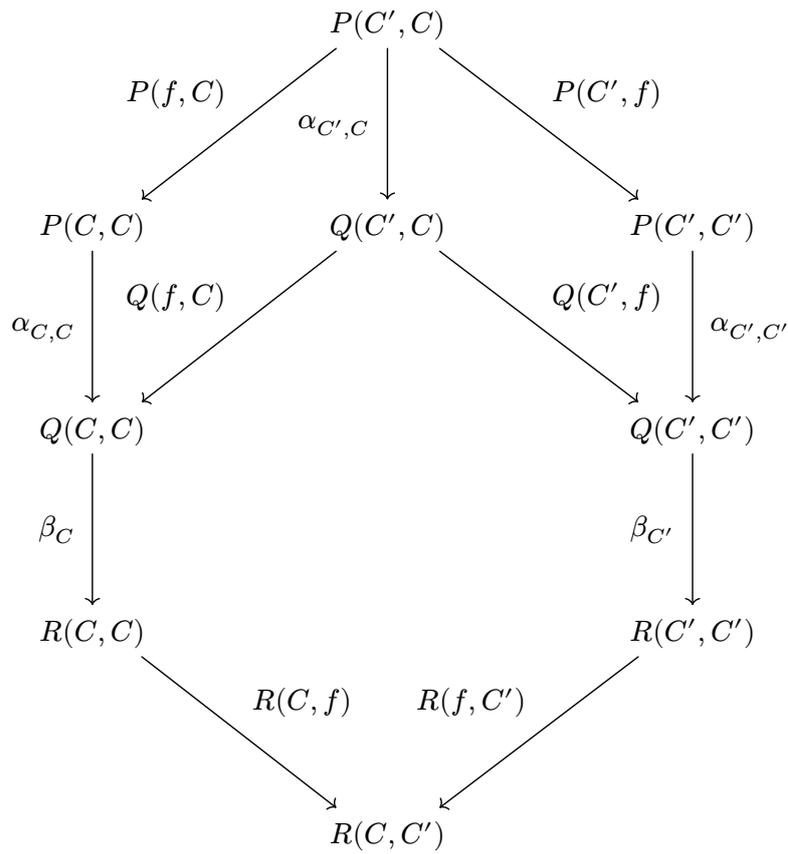
which does not commute: the left morphism is the constant ff morphism, while the right one is the constant tt morphism.

2. Consider $\alpha : P \rightrightarrows Q$ be a dinatural transformation, and $\beta : Q \rightrightarrows R$ be a natural transformation. Let $(\beta \circ \alpha)_C = \beta_{C,C} \circ \alpha_C$. Let us check that this indeed forms a dinatural transformation. Let $f : C \rightarrow C'$ be a morphism in \mathcal{C} . The following diagram commutes, as

the upper hexagon commutes by dinaturality of α , and the two lower squares commute by naturality of β



Consider now a natural transformation $\alpha : P \Rightarrow Q$ and a dinatural transformation $\beta : Q \rightrightarrows R$. Let $(\beta \circ \alpha)_C = \beta_C \circ \alpha_{C,C}$. Let us check that this indeed forms a dinatural transformation. Let $f : C \rightarrow C'$ be a morphism in \mathcal{C} . The following diagram commutes, as the lower hexagon commutes by dinaturality of β , and the upper squares commute by naturality of α .



□

Exercise 1.3

Let \mathcal{C} and \mathcal{D} be two non-empty categories, and $D : \mathcal{D}$ be an object of \mathcal{D} .

1. compute $\int_{\mathcal{C}; \mathcal{C}} D$
2. compute $\int_{\mathcal{C}; \mathcal{C}} D$

♡

Proof.

1. Consider an other object X with $\alpha : X \rightrightarrows D$. For any two objects $C, C' : \mathcal{C}$, and a morphism $f : C \rightarrow C'$, we have have $\alpha_C = \alpha_{C'}$ from the commutation of this diagram

$$\begin{array}{ccc}
& X & \\
& \swarrow \text{id}_X & \searrow \text{id}_X \\
X & & X \\
\downarrow \alpha_C & & \downarrow \alpha_{C'} \\
D & & D \\
& \swarrow \text{id}_D & \searrow \text{id}_D \\
& D &
\end{array}$$

α is therefore constant on each connected component of \mathcal{C} , hence, if we denote by $c(\mathcal{C})$ the connected components of \mathcal{C} , we have, if the end exists,

$$\begin{aligned}
\mathcal{D}\left(X, \int_{C:\mathcal{C}} D\right) &\cong \mathcal{D}(c(\mathcal{C}) \otimes X, D) \\
&\cong \mathcal{D}(X, D)^{c(\mathcal{C})} \\
&\cong \mathcal{D}(X, c(\mathcal{C}) \pitchfork D)
\end{aligned}$$

and hence, by the Yoneda lemma, we find out that we must have

$$\int_{C:\mathcal{C}} D = c(\mathcal{C}) \pitchfork D$$

The proof of this is easy: consider the wedge $c(\mathcal{C}) \pitchfork D$ and $\alpha : c(\mathcal{C}) \pitchfork D \Rightarrow D$ defined, as shown earlier, on each connected component $s : c(\mathcal{C}) \pitchfork D$ by

$$\alpha_s = \pi_s$$

The rest is immediate, by noting that $\int_{C:\mathcal{C}} D$ has the same universal property as $c(\mathcal{C}) \pitchfork D$.

2. By the same argument as above, a cowedge on D is the same thing as an object X and, for each connected component $s : c(\mathcal{C})$, a morphism $\alpha_s : D \rightarrow X$. Hence

$$\begin{aligned}
\mathcal{D}\left(\int^{C:\mathcal{C}} D, X\right) &\cong \mathcal{D}(D, s(\mathcal{C}) \pitchfork X) \\
&\cong \mathcal{D}(c(\mathcal{C}) \otimes D, X)
\end{aligned}$$

and thus

$$\int^{C:\mathcal{C}} D = c(\mathcal{C}) \otimes D$$

□

Exercise 1.4

Show that extranatural transformations compose accordingly to these rules:

- (stalactites) Let $F, G : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ be functors. If $\alpha : F \Rightarrow G$ is a natural transformation, and $\beta_X : G(X, X) \rightarrow H$ is extranatural in X (for some $H : \mathcal{D}$), then

$$\beta_X \circ \alpha_{X,X} : F(X, X) \rightarrow H$$

is extranatural in X .

- (stalagmites) Let $G, H : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$. If $\alpha_X : F \rightarrow G(X, X)$ is extranatural in X (for some object $F : \mathcal{D}$), and $\beta : G \Rightarrow H$ is a natural transformation, then

$$\beta_{X,X} \circ \alpha_X : F \rightarrow H(X, X)$$

is extranatural in X .

- (yanking) Let $F, H : \mathcal{C} \rightarrow \mathcal{D}$, and let $G : \mathcal{C} \times \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ be a functor. If $\alpha_{X,Y} : F(Y) \rightarrow G(X, X, Y)$ is natural in Y and extranatural in X , and if $\beta_{X,Y} : G(X, Y, Y) \rightarrow H(X)$ is natural in X and extranatural in Y , then

$$\beta_{X,X} \circ \alpha_{X,X} : F(X) \rightarrow H(X)$$

is natural in X .

Express these laws as equalities between suitable string diagrams (explaining also the genesis of the names ‘stalactite’ and ‘stalagmite’).



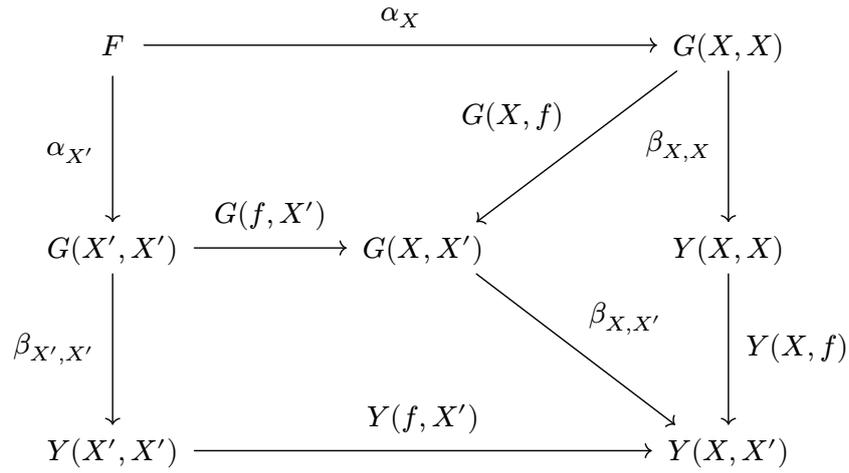
Proof.

- We only need to check a single commutativity diagram: let $X, X' : \mathcal{C}$, and $f : X \rightarrow X'$ a morphism in \mathcal{C}

$$\begin{array}{ccccc}
 F(X', X) & \xrightarrow{F(X', f)} & & & F(X', X') \\
 \downarrow F(f, X) & \searrow \alpha_{X', X} & & & \downarrow \alpha_{X', X'} \\
 F(X, X) & & G(X', X) & \xrightarrow{G(X', f)} & G(X', X') \\
 \downarrow \alpha_{X, X} & \swarrow G(f, X) & & & \downarrow \beta_{X'} \\
 G(X, X) & \xrightarrow{\beta_X} & & & H
 \end{array}$$

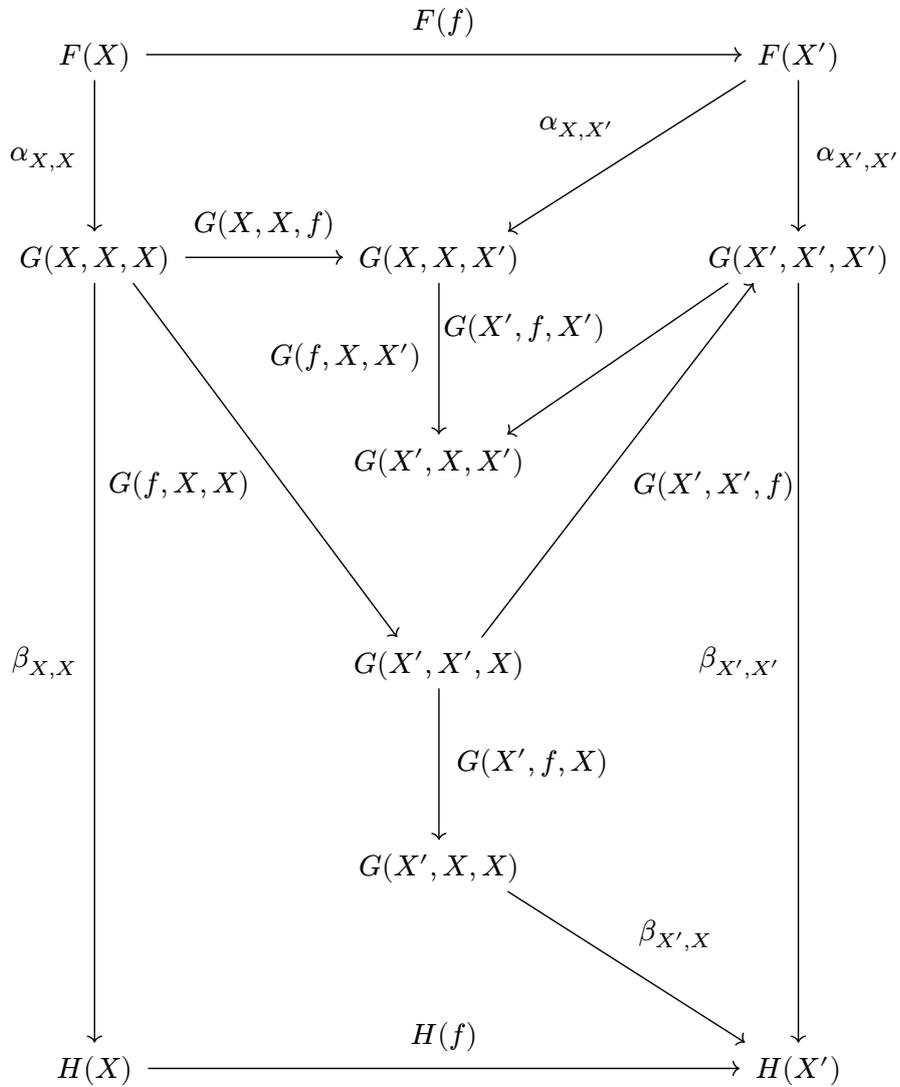
The upper-right square commutes by naturality of α , the left one too, and the lower-right square commutes by extranaturality.

- Again, we only need to check a single commutativity diagram: let $X, X' : \mathcal{C}$ and $f : X \rightarrow X'$ a morphism



the upper-left square commutes by extranaturality of α , while the lower-left and rightmost squares commute by naturality of β .

- Let $X, X' : \mathcal{C}$ be two objects, and $f : X \rightarrow X'$ a morphism between them.



□

Exercise 1.5

Prove that dinaturality is strictly more general than extranaturality.



Proof. Let $\mathbf{2}$ be the walking arrow category, and $S, T : \mathbf{2}^{\text{op}} \times \mathbf{2} \rightarrow \mathbf{Set}$ be defined by

$$\begin{array}{ccc}
 \begin{array}{ccc} \{1\} & \xrightarrow{c_1} & \{1, 2\} \\ \text{id}_{\{1\}} \downarrow & S & \sigma \downarrow \\ \{1\} & \xrightarrow{c_2} & \{1, 2\} \end{array} &
 \begin{array}{ccc} (1, 0) & \longrightarrow & (1, 1) \\ \downarrow & & \downarrow \\ (0, 0) & \longrightarrow & (0, 1) \end{array} &
 \begin{array}{ccc} \{1\} & \xrightarrow{\text{id}_{\{1\}}} & \{1\} \\ \text{id}_{\{1\}} \downarrow & T & c_2 \downarrow \\ \{1\} & \xrightarrow{c_2} & \{1, 2\} \end{array}
 \end{array}$$

where c_i is the constant function at i , at $\sigma : \{1, 2\} \rightarrow \{1, 2\}$ is the non-trivial permutation.

Consider the transformation

$$\begin{aligned}
 \alpha : T &\Rightarrow S \\
 0 &\mapsto \text{id}_{\{1\}} \\
 1 &\mapsto \{1\} \subseteq \{1, 2\}
 \end{aligned}$$

it is dinatural, as the following diagram clearly commutes

$$\begin{array}{ccccc}
 & & T(1, 0) = \{1\} & & \\
 & \swarrow & & \searrow & \\
 & T(1 > 0, 0) = \text{id}_{\{1\}} & & T(1, 0 < 1) = \text{id}_{\{1\}} & \\
 & & T(0, 0) = \{1\} & & T(1, 1) = \{1\} \\
 & \downarrow & & & \downarrow \\
 \alpha_0 = \text{id}_{\{1\}} & & & & \alpha_1 = \{1\} \subseteq \{1, 2\} \\
 & & S(0, 0) = \{1\} & & S(1, 1) = \{1, 2\} \\
 & \searrow & & \swarrow & \\
 & S(0, 0 < 1) = c_2 & & S(0 < 1, 1) = \sigma & \\
 & & S(0, 1) = \{1, 2\} & &
 \end{array}$$

Yet, it is not extranatural if we see it as a transformation $T \Rightarrow S$ where $T : \mathbf{1} \times \mathbf{2}^{\text{op}} \times \mathbf{2} \rightarrow \mathbf{Set}$ and $S : \mathbf{1} \times \mathbf{2} \times \mathbf{2}^{\text{op}} \rightarrow \mathbf{Set}$. Indeed, assume it is, then take $B = 0$, $B' = 1$ and $C = 1$. We have

$$\begin{array}{ccc}
 & T(1, 0 < 1) = \text{id}_{\{1\}} & \\
 T(1, 0) = \{1\} & \xrightarrow{\quad} & T(1, 1) = \{1\} \\
 \downarrow & & \downarrow \\
 T(1 > 0, 0) = \text{id}_{\{1\}} & & \alpha_1 = \{1\} \subseteq \{1, 2\} \\
 \downarrow & & \downarrow \\
 T(0, 0) = \{1\} & \xrightarrow{\alpha_{0,1}} & S(1, 1) = \{1, 2\}
 \end{array}$$

So $\alpha_{0,1} = \{1\} \subseteq \{1, 2\}$. Let's now take $B = 0$, $C = 0$ and $C' = 1$.

$$\begin{array}{ccc}
 T(0, 0) = \{1\} & \xrightarrow{\alpha_0 = \text{id}_{\{1\}}} & S(0, 0) = \{1\} \\
 \downarrow \alpha_{0,1} = \{1\} \subseteq \{1, 2\} & & \downarrow S(0, 0 < 1) = c_2 \\
 S(1, 1) = \{1, 2\} & \xrightarrow{S(1 < 0, 1) = \sigma} & S(0, 1) = \{1, 2\}
 \end{array}$$

□

Exercise 1.6

Let $n : \mathbb{N}$, and, for $i : \{1, \dots, n\}$, \mathcal{A}_i be a category with an initial and a terminal object. Let \mathcal{C} be a category, and $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \prod_{i=1}^n \mathcal{A}_i$ be a functor such that every $\int_C \pi_i(F(C, C))$ (resp. $\int^C \pi_i(F(C, C))$) exists. Show that

$$\int_C F(C, C) = \left(\int_C \pi_1(F(C, C)), \dots, \int_C \pi_n(F(C, C)) \right)$$

resp.

$$\int^C F(C, C) = \left(\int^C \pi_1(F(C, C)), \dots, \int^C \pi_n(F(C, C)) \right)$$

♥

Proof. Let $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$, $i : \{1, \dots, n\}$, $X : \mathcal{A}_i$ and $\hat{X} = (0, \dots, X, \dots, 0)$.

$$\begin{aligned}
 \mathcal{A}_i \left(X, \pi_i \left(\int_C F(C, C) \right) \right) &\cong \mathcal{A} \left(\hat{X}, \int_C F(C, C) \right) \\
 &\cong \int_C \mathcal{A}(\hat{X}, F(C, C)) \\
 &\cong \int_C \mathcal{A}_i(X, \pi_i(F(C, C))) \\
 &\cong \mathcal{A}_i \left(X, \int_C \pi_i(F(C, C)) \right)
 \end{aligned}$$

Thus, by the Yoneda lemma, we have

$$\pi_i \left(\int_C F(C, C) \right) \cong \int_C \pi_i(F(C, C))$$

and thus

$$\int_C F(C, C) \cong \left(\int_C \pi_1(F(C, C)), \dots, \int_C \pi_n(F(C, C)) \right)$$

□

Exercise 1.7

Let \mathcal{D} be a category. Show that the end of a functor $T : \Delta[1]^{\text{op}} \times \Delta[1] \rightarrow \mathcal{D}$ is the following pullback

$$\begin{array}{ccc}
 \int_{i:\Delta[1]} T(i, i) & \longrightarrow & T(0, 0) \\
 \downarrow & & \downarrow \\
 & T(0, 0 < 1) & \\
 & & \\
 T(1, 1) & \xrightarrow{T(1 > 0, 0)} & T(0, 1)
 \end{array}$$

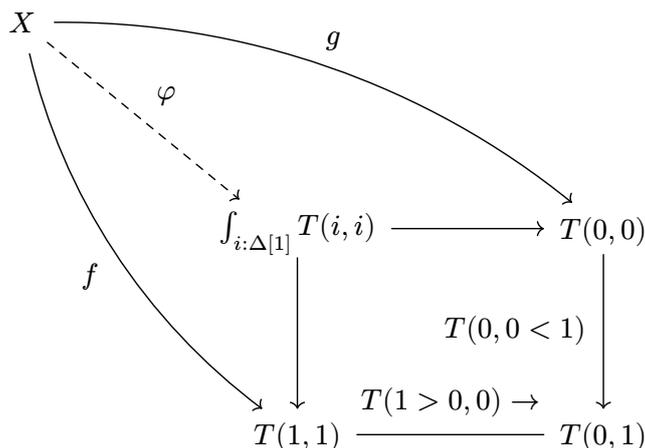
Proof. Firstly, the diagram commutes by definition of a wedge. Consider now $X : \mathcal{D}$ with $f : X \rightarrow T(1, 1)$ and $g : X \rightarrow T(0, 0)$ such that the following diagram commutes

$$\begin{array}{ccc}
 X & \xrightarrow{g} & T(0, 0) \\
 \searrow f & & \downarrow \\
 & \int_{i:\Delta[1]} T(i, i) & \longrightarrow T(0, 0) \\
 & \downarrow & \\
 & T(1, 1) & \xrightarrow{T(1 > 0, 0)} T(0, 1)
 \end{array}$$

This defines exactly a wedge $\alpha : X \rightrightarrows T$, where $\alpha_0 = g$ and $\alpha_1 = f$. Hence, there exists a unique $\varphi : X \rightarrow \int_i T(i, i)$ making the following diagram commute

$$\begin{array}{ccc}
 X & \xrightarrow{\varphi} & \int_{i:\Delta[1]} T(i, i) \\
 \parallel & & \parallel \\
 & & T
 \end{array}$$

That is, the following diagram commutes



□

Exercise 1.8

Let G be a topological group, and $\text{Sub}(G)$ the set of its subgroups partially ordered by inclusion; let X be a G -space, i.e. a topological space with a continuous action $X \times X \rightarrow X$.

We can define two functors $\text{Sub}(G) \rightarrow \mathbf{Top}$, sending $(H \leq G) \mapsto G/H$ (this is a covariant functor, and G/H has the induced quotient topology as a space; there is no need for H to be normal) and $(H \leq G) \mapsto X^H$ (the sub of H -fixed points for the action; this is a contravariant functor).

- Compute the coend

$$\circ_G(X) = \int^{H \leq G} X^H \times G/H$$

in the category \mathbf{Top} of topological spaces, if $G = \mathbb{Z}/2$ has the discrete topology.

- Give a general rule for computing $\circ_G(X)$ when $G = \mathbb{Z}/n\mathbb{Z}$ is cyclic with n elements.
- Let instead $\text{Orb}(G)$ be the orbit category of subgroups of G , whose objects are subgroups but $\text{hom}(H, K)$ contains G -equivariant maps $G/H \rightarrow G/K$. Let again X^- and $G/-$ define the same functors, now with different action on arrows. Prove that

$$\int^{H: \text{Orb}(G)} X^H \times G/H \cong X$$

- Let $E|F$ be a field extension, and $\{H \leq \text{Gal}(E|F)\}$ the partially ordered set of subgroups of the Galois group of extensions. Compute (in the category of rings) the coend

$$\int^H E^H \times \text{Gal}(H|F)$$

♡

Exercise 1.9

Dualise the construction in 1.2, to obtain a characterisation for the coend $\int^C F(C, C)$, characterised as the coequaliser of a pair (F^*, F_*) as in

$$\coprod_{C \rightarrow C'} F(C', C) \rightrightarrows \coprod_{C: \mathcal{C}} F(C, C)$$

Exercise 1.10

Find an alternative proof that natural transformations can be written as an end (see 1.4.1), using the characterisation of $\int_C \mathcal{D}(F(C), G(C))$ as an equiser in 1.2.4: as a subset of $\prod_{C: \mathcal{C}} \mathcal{D}(FC, GC)$, is precisely the subset of natural transformations $\{\tau_C : FC \rightarrow GC \mid \forall f : C \rightarrow C', Gf \circ \tau_C = \tau_{C'} \circ Ff\}$

Exercise 1.11

What is the co/end of the identity functor

$$\int_C (C, C) \quad \int^C (C, C)$$

Use the bare definition. Use the characterisation of co/ends as co/limits; feel free to invoke Exercise 1.6.

Exercise 1.12

A set of objects $S \subset C$ regarded as a full subcategory, *finitely generates* a category \mathcal{C} if for each object $X : \mathcal{C}$, and each arrow $f : S \rightarrow C$ from $S : \mathcal{S}$ there is a factorisation

$$S \xrightarrow{g} \prod_{i=1}^n S_i \xrightarrow{h_C} C$$

where h_C is an epimorphism and $\{S_1, \dots, S_n\} \subset \mathcal{S}$ (n depends on C and f).

Suppose $T : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ is a functor, finitely continuous in both variables, and \mathcal{C} is finitely generated by S . Then if we denote $T|_{\mathcal{S}} : \mathcal{S}^{\text{op}} \times \mathcal{S} \rightarrow \mathbf{Set}$ the restriction, we have an isomorphism

$$\int^{C: \mathcal{S}} T|_{\mathcal{S}}(C, C) \cong \int^{C: \mathcal{C}} T(C, C)$$

induced by a canonical arrow

$$\int^{C: \mathcal{S}} T'(C, C) \longrightarrow \int^{C: \mathcal{C}} T(C, C)$$

Exercise 1.13

Let $F \dashv U : \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction, and $G : \mathcal{D}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{E}$ a functor; then there is an isomorphism

$$\int^C G(FC, C) \cong^D G(G, UD)$$

Show that the converse of this result is true: if the above isomorphism is true for any G and natural therein, then there is an adjunction $F \dashv U$.

Exercise 1.14