Notes on categorical semantics of logic

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Introduction

The idea that motivates the current note is that every class of theories can be expressed as a category of sufficiently structure rich categories. For instance, when a category has a terminal object 1, one can see objects of this category as collection of their elements, where an *element of an object A* is simply a morphism $1 \to A$. This suggests that, to be able to speak categorically of theories that exhibit constants, one just needs the "ambient category" to have a terminal object.

Taking this idea further, one can see classes of categories having sufficiently rich structure as those categories that can be taken as "ambient" categories in which to develop a given theory. We will begin by investigating categories in which one can take limits of a given shape. Each class of shapes will produce a class of theories one can speak of, but, most importantly, the categorical structure of the class of shapes, **Shp**, will give us a way to *mix* theories of different kind.

For instance, one can see categories as "higher sets", that is, a (large) set equiped with morphisms. This suggests that in order to get a "higher" version of a theory that usually takes place in sets, one can perform the theory with categories rather than sets. For instance, given a field $\mathbb{K}$, one can consider vector spaces over $\mathbb{K}$, but also vector spaces over $\mathbb{K}$ internal to **Cat**. However, we can also recover the same models by considering models of a *product theory* (the product of then theory of vector spaces, and the theory of categories). One therefore derives $\mathbf{2\text{-}Vect}_{\mathbb{K}} \coloneqq \mathbf{Vect}_{\mathbb{K}} \times \mathbf{Cat}$, the theory of 2-vector spaces. The main interest here of doing so is that one recovers the ability to interpret this theory in different categories.

Chapter 1. Shape calculus

For this section, consider $\mathcal{F}$ a class of (small) categories. Elements of $\mathcal{F}$ are called *shapes*.

Definition 1.1 ($\mathcal{F}$-complete category)

A $\mathcal{F}$-complete category is a category that has all limits of all shapes $\mathcal{S} : \mathcal{F}$. ♣

1.1. Category of shapes

At first, we will consider a fixed $\mathcal{F}$, and develop the theory around it. For instance, we will see how to compute the product of two $\mathcal{F}$ theories. However, this doesn't inform us on how to do the product of an $\mathcal{F}$ theory with a $\mathcal{F}'$ theory: for instance, the theory of higher vector spaces will be built as a product of the theory of vector spaces with the theory of higher categories.

Definition 1.1.1 (Category of shapes)

Let **Shp** be the preorder category whose objects are collection of categories, and there is a morphism from $\mathcal{F}$ to $\mathcal{F}'$ if every $\mathcal{F}'$-complete category $\mathcal{C}$ is also $\mathcal{F}$-complete. ♣

1.2. $\mathcal{F}$-complete categories

Let us note $\mathcal{F}\mathbf{Cat}$ the category of $\mathcal{F}$-complete categories with morphisms $\mathcal{S}$-continuous functors for every $\mathcal{S} : \mathcal{F}$.

Example. The category of cartesian categories **CCat** is a category of $\mathcal{F}$-complete categories for

$$\mathcal{F} := \{\mathbf{1}, 2\}$$

where 2 is the discrete category with two elements.

Example. The category of (finitely) complete categories is a category of $\mathcal{F}$-complete category, by taking $\mathcal{F}$ to be the class of all (finite) categories.

Remark

Multiple different $\mathcal{F}$ can lead to the same category $\mathcal{F}\mathbf{Cat}$. For instance, equalizers and finite products are enough to have all finite limits; similarly, having pullbacks and a terminal object is enough to have all finite limits.

The cornerstone of the theory of $\mathcal{F}$-complete categories is stated as follows.

Theorem 1.2.1

$\mathcal{F}\mathbf{Cat}$ is cartesian closed. ♡

Proof. First of all, $\mathcal{F}\mathbf{Cat}$ has a terminal object: $\mathbf{1}$. Indeed, $\mathbf{1}$ is complete. Furthermore, every functor $\mathcal{C} \xrightarrow{!} \mathbf{1}$ is complete. Let us now consider $\mathcal{C}, \mathcal{D} : \mathcal{F}\mathbf{Cat}$ two categories, and let us show that $\mathcal{C} \times \mathcal{D}$ is in $\mathcal{F}\mathbf{Cat}$. Let $\mathcal{S} : \mathcal{F}$ be a shape, and $F : \mathcal{S} \to \mathcal{C} \times \mathcal{D}$ a functor. We can write $F = \langle F_1, F_2 \rangle$. Let us show that $(\lim F_1, \lim F_2)$ is a limit of F. This stems from the following (natural) identities, for $(X, Y) : \mathcal{C} \times \mathcal{D}$:

$$\begin{aligned}(\mathcal{C} \times \mathcal{D})((X,Y),(\lim F_1, \lim F_2)) &= \mathcal{C}(X, \lim F_1) \times \mathcal{D}(Y, \lim F_2) \\ &\cong (X \Rightarrow F_1) \times (Y \Rightarrow F_2) \\ &\cong (X,Y) \Rightarrow F_1 \times F_2\end{aligned}$$

By the particular choice of limits in $\mathcal{C} \times \mathcal{D}$ that we have exhibited, we can immediately deduce that $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ are continuous for shapes in $\mathcal{F}$. It is immediate that $\mathcal{C} \times \mathcal{D}$ is a cartesian product of $\mathcal{C}$ with $\mathcal{D}$.

Let us finally check that $\mathcal{F}\mathbf{Cat}$ is closed. Consider $\mathcal{C}$ and $\mathcal{D}$ be two categories in $\mathcal{F}\mathbf{Cat}$, define $\mathcal{D}^{\mathcal{C}}$ be the category of $\mathcal{F}$-continuous functors, with natural transformations as morphisms. Since limits of shapes in $\mathcal{F}$ exist in $\mathcal{D}$, they exist in $\mathcal{D}^{\mathcal{C}}$. Since limits are computed pointwise, the same proof that show that **Cat** is closed works to show that $\mathcal{D}^{\mathcal{C}}$ is an internal hom. □

Definition 1.2.2 (Shape functor)

There is a functor

$$\begin{aligned} -\mathbf{Cat} : \quad \mathbf{Shp}^{\mathsf{op}} &\longrightarrow \mathbf{Cat} \\ \mathcal{F} &\longmapsto \mathcal{F}\mathbf{Cat} \\ \mathcal{F} \leq \mathcal{F}' &\longmapsto \mathcal{F}'\mathbf{Cat} \subseteq \mathcal{F}\mathbf{Cat} \end{aligned}$$

♣

Theorem 1.2.3

Let $\mathcal{F} \leq \mathcal{F}'$, the functor $(\mathcal{F} \leq \mathcal{F}')\mathbf{Cat} : \mathcal{F}'\mathbf{Cat} \to \mathcal{F}\mathbf{Cat}$ has a left adjoint. Let us call this left adjoint $F_{\mathcal{F}}^{\mathcal{F}'} : \mathcal{F}\mathbf{Cat} \to \mathcal{F}'\mathbf{Cat}$.

♡

Proof. See [1] Let $\mathcal{C}$ be a $\mathcal{F}$-complete category. Let us first show that we can freely add all limits of shape $\mathcal{F}'$ in $\mathcal{C}$. Define $\hat{\mathcal{F}}'$ be the greatest class of shapes equivalent to $\mathcal{F}'$, that is,

$$\hat{\mathcal{F}}' := \bigcup \left\{ \tilde{\mathcal{F}} : \mathbf{Shp} \mid \tilde{\mathcal{F}} \leq \mathcal{F}' \wedge \mathcal{F}' \leq \tilde{\mathcal{F}} \right\}$$

Consider the category $\mathcal{C}_{\mathcal{F}'}$, where objects are of the form $F : \mathcal{S} \to \mathcal{C}$ where $\mathcal{S} : \hat{\mathcal{F}}'$. Intuitively, such a diagram represents its added limit in $\mathcal{C}$. Hence, we must have, for $F : \mathcal{S} \to \mathcal{C}$ and $G : \mathcal{S}' \to \mathcal{C}$,

$$\mathcal{C}_{\mathcal{F}'}(F, G) := \lim_{s' : \mathcal{S}'} \operatorname*{colim}_{s : \mathcal{S}} \mathcal{C}(F(s), G(s'))$$

This is a category. Indeed, for $F : \mathcal{S} \to \mathcal{C}$, we have an identity given by the following family of elements

If I have a natural transformation $\alpha : F \Rightarrow G : \mathcal{C} \to \mathcal{D}$, and a functor $H : \mathcal{E} \to \mathcal{C}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{* : 1, s : \mathcal{S} \vdash_{\mathbf{Set}} \mathrm{id}_{F(s)} : \mathcal{C}(F^{\mathsf{op}}(s), F(s))}\,\text{Id-I}}{* : 1 \vdash_{\mathbf{Set}^{\mathcal{S}}} \mathrm{id} * F : \mathcal{C}(F^{\mathsf{op}}(-), F(-))}\,\text{nat}}{* : 1 \vdash_{\mathbf{Set}^{\mathcal{S}}} \_ : \mathrm{colim}_{s : \mathcal{S}}\, \mathcal{C}(F^{\mathsf{op}}(s), F(-))}\,- * \iota}{* : 1 \vdash_{\mathbf{Set}} \_ : \lim_{s' : \mathcal{S}} \mathrm{colim}_{s : \mathcal{S}}\, \mathcal{C}(F^{\mathsf{op}}(s), F(s'))}\,\text{lim-I}}{\vdash_{\mathbf{Set}} \_ : \lim_{s' : \mathcal{S}} \mathrm{colim}_{s : \mathcal{S}}\, \mathcal{C}(F^{\mathsf{op}}(s), F(s'))}\,-(*)$$

$$\left(s, \mathrm{id}_{F(s)}\right)_{s : \mathcal{S}} \in \lim_{s : \mathcal{S}} \operatorname*{colim}_{s : \mathcal{S}} \mathcal{C}(F(s), F(s))$$

$$\mathcal{C}_{\mathcal{F}'}(F, F)$$

so, for $s : \mathcal{S}^{\mathsf{op}}$, and $s' : \mathcal{S}$, we have to find an element of

$$\mathcal{C}(F^{\mathsf{op}}(s), F(s'))$$

□

Proposition 1.2.4

For $\mathcal{F} \leq \mathcal{F}' \leq \mathcal{F}''$, we have

$$F^{\mathcal{F}}_{\mathcal{F}''} \cong F^{\mathcal{F}}_{\mathcal{F}'} \circ F^{\mathcal{F}'}_{\mathcal{F}''}$$

♠

Proof. TODO this stems by uniqueness of left adjoints. □

1.3. $\mathcal{F}$ theory

Definition 1.3.1 ($\mathcal{F}$ theory)

A $\mathcal{F}$ theory $\mathcal{T}$ is an $\mathcal{F}$-complete category.

♣

A morphism of $\mathcal{F}$ theory is simply a morphism of $\mathcal{F}$-complete category. In fact, there is no technical distinction between the category of $\mathcal{F}$ theories and $\mathcal{F}\mathbf{Cat}$. However, in what follows, we want to see those categories as *theories*, in the sense that they admit a (meta) theory of models.

1.4. Model of a $\mathcal{F}$ theory

Let $\mathcal{U} : \mathcal{F}\mathbf{Cat}$ be a category, called in this context a *universe*, and $\mathcal{T}$ be a $\mathcal{F}$ theory.

Definition 1.4.1 (Model of a theory)

A *model* M *of* $\mathcal{T}$ *in* $\mathcal{U}$ is a morphism $\mathcal{T} \to \mathcal{U}$, that is, a functor that is continuous with respect to all limits of shapes in $\mathcal{F}$.

♣

Definition 1.4.2 (Morphism of models)

Given M and M' be two models of $\mathcal{T}$, a *morphism from* M *to* M' is a natural transformation $M \Rightarrow M'$.

♣

Definition 1.4.3 (Category of models)

For a given theory $\mathcal{T}$, we define its category of models (in $\mathcal{U}$)

$$\mathbf{Mod}_{\mathcal{T}}(\mathcal{U}) := \mathcal{F}\mathbf{Cat}(\mathcal{T}, \mathcal{U})$$

♣

Remark

We are often primarily interested in models in **Set** (which belongs to every $\mathcal{F}$Cat). However, being able to change the category in which we interpret are theory will be an other important tool to make compute describe theories later on.

1.5. Monad on an $\mathcal{F}$ theory

For this section, fix $\mathcal{T}$ a $\mathcal{F}$ theory, and $\mathcal{U}$ be a universe, which has all limits shaped by $\mathcal{T}$.

Definition 1.5.1 (Free algebra monad)

Let us note $F_{\mathcal{T}} : \mathcal{U} \to \mathcal{U}$ the monad on $\mathcal{U}$ derived from the following adjunction

$$\mathcal{U} \underset{\lim}{\overset{\Delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{Mod}_{\mathcal{T}}(\mathcal{U})$$

which we call the *free algebra monad over* $\mathcal{U}$.

Remark

If $\mathcal{U}$ is complete, then in particular it has all limits of shape $\mathcal{T}$, so we can always build the free algebra monad over it.

Example. **Set** is complete. Therefore, every theory induces a free algebra monad over the universe **Set**. We will see how to recover usual monads on **Set** with this construction.

Proposition 1.5.2

The adjunction $\Delta \dashv \lim$ is monadic.

Proof. Consider the category $\mathcal{U}^{F_{\mathcal{T}}}$ of algebras of the monad $F_{\mathcal{T}}$. There is a functor

$$G : \mathbf{Mod}_{\mathcal{T}}(\mathcal{U}) \longrightarrow \mathcal{U}^{F_{\mathcal{T}}}$$

given by, for any element $M : \mathbf{Mod}_{\mathcal{T}}(\mathcal{U})$,

$$(\lim M, \lim \varepsilon_M)$$

□

Finish this proof.

Corollary 1.5.2.1

The category of algebras of the free algebra monad $F_{\mathcal{T}}$ is isomorphic to the category of models of $\mathcal{T}$.

Chapter 2. Quantifier-free first order logic

2.1. Algebraic theories

For this section, we will consider categories with finite products, that is, $\mathcal{F} = \{0, 2\}$.

Definition 2.1.1 (Multi-sorted Lawvere theory)

A $\mathcal{F}$ theory $\mathcal{T}$ is called a *multi-sorted Lawvere theory.* ♣

Definition 2.1.2 (Category of models of T)

Given an algebraic theory T, its models, equiped with morphisms of models, form a category $\mathbf{Mod}_T$ of models of T. ♣

2.2. Horn theories

2.3. Essentially algebraic theories

Chapter 3. Higher order logic

Chapter 4. Lawvere Theories

Definition 4.1 (Lawvere theory)

A *(generalized) Lawvere theory* is a cartesian category. ♣

Definition 4.2 (Model of a Lawvere theory)

A *model of a lawvere theory* $\mathcal{T}$ *in a cartesian category* $\mathcal{C}$ is a cartesian functor $M : \mathcal{T} \to \mathcal{C}$. ♣

Definition 4.3 (Category of models of a Lawvere theory)

Given a Lawvere theory $\mathcal{T}$, we define its *category of models* $\mathbf{Mod}_{\mathcal{T}}(\mathcal{C})$ *in a cartesian category* $\mathcal{C}$ as

$$\mathbf{Mod}_{\mathcal{T}}(\mathcal{C}) = [\mathcal{T}, \mathcal{C}]$$

♣

Definition 4.4 (Morphism of Lawvere theory)

Given two Lawvere theories $\mathcal{T}$ and $\mathcal{T}'$, a *morphism of Lawvere theories* $\mathcal{T} \to \mathcal{T}'$ is a cartesian functor $F : \mathcal{T} \to \mathcal{T}'$. ♣

Proposition 4.5

Given two Lawvere theories $\mathcal{T}$ and $\mathcal{T}'$, and a morphism $F : \mathcal{T} \to \mathcal{T}'$. Let $\mathcal{C}$ be a cartesian category. There exists a cartesian functor $\mathbf{Mod}_F(\mathcal{C}) : \mathbf{Mod}_{\mathcal{T}'}(\mathcal{C}) \to \mathbf{Mod}_{\mathcal{T}}(\mathcal{C})$. ♠

Proof. Let $M : \mathcal{T}' \to \mathcal{C}$, we have $M \circ F : \mathcal{T} \to \mathcal{C}$. Furthermore, given a natural transformation $\alpha : M_1 \to M_2$, we have

$$\alpha * F : M_1 \circ F \to M_2 \circ F$$

which is functorial: $\mathrm{id}_M * F = \mathrm{id}_{M \circ F}$ and $(\alpha \circ \beta) * F = (\alpha * F) \circ (\beta * F)$.

Let $M_1, M_2 : \mathcal{T} \to \mathcal{C}$ be two models. $(M_1 \times M_2) \circ F = (M_1 \circ F) \times (M_2 \circ F)$ by definition, since limits are computed pointwise. Similarly, $\pi_i * F = \pi_i : (M_1 \circ F) \times (M_2 \circ F) \to M_i \circ F$. □

Proposition 4.6

$\mathbf{Mod}\_(\mathcal{C}) : \mathbf{CCat}^{\mathrm{op}} \to \mathbf{CCat}$ est un foncteur contravariant continu. ♠

Proof. C'est exactement le foncteur $\mathbf{CCat}(-, \mathcal{C})$, c'est donc bien un foncteur contravariant, et il est bien continu. □

Example. Consider an algebraic theory T. We can build its Lawvere theory $\mathcal{T}$ as follows: objects of $\mathcal{T}$ are words of S, noted $s_1 \times \cdots \times s_n$. A morphism from a $\prod_{i=1}^n s_i \to \prod_{i=1}^m s'_i$ is a tuple $(t_1, \ldots, t_m)$ where each t_i is a term in the context $\Gamma \coloneqq x_1 : s_1, \ldots, x_n : s_n$ on the language $\mathcal{L}$, quotiented by the equivalence relation $t_i \sim t'_i$ if $T \vdash t_i = t'_i$.

We have an important result:

$$\mathbf{Mod}_{\mathcal{T}}(\mathbf{Set}) \cong \mathbf{Mod}_T$$

Theorem 4.7

Consider $\mathcal{T}_1$ and $\mathcal{T}_2$ two Lawvere theories. A $\mathcal{T}_1$-model in the category of $\mathcal{T}_2$-models is the same thing as a $\mathcal{T}_2$-model in the category of $\mathcal{T}_1$-models. In fact, there is a (natural) isomorphism, for any cartesian category $\mathcal{C}$,

$$\mathbf{Mod}_{\mathcal{T}_1}\left(\mathbf{Mod}_{\mathcal{T}_2}(\mathcal{C})\right) \cong \mathbf{Mod}_{\mathcal{T}_2}\left(\mathbf{Mod}_{\mathcal{T}_1}(\mathcal{C})\right)$$

♡

Proof. □

Theorem 4.8

Two Lawvere theories $\mathcal{T}_1$ and $\mathcal{T}_2$ such that, for any cartesian category $\mathcal{C}$, $\mathbf{Mod}_{\mathcal{T}_1}(\mathcal{C}) \cong \mathbf{Mod}_{\mathcal{T}_2}(\mathcal{C})$ naturally in $\mathcal{C}$, then $\mathcal{T}_1 \cong \mathcal{T}_2$.

♡

Proof. This stems directly from the Yoneda lemma. □

Bibliography

[1] E. Beurier, D. Pastor, and R. Guitart, “Presentations of Clusters and Strict Free-Cocompletions,” *Theory and Applications of Categories*, vol. 36, no. 17, pp. 492–513, Aug. 2021, [Online]. Available: https://imt-atlantique.hal.science/hal-03329332