Syntax and Semantics of a Linear Dependent Type Theory

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Contents

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If it was so, it might be; and if it were so, it would be,
but as it isn't, it ain't. That's logic.
— Tweedledee, *about LDTT*

1. Summary

1.1. General Context

Over the last decades, proof mechanization has become a highly focused problem in logic. By proof mechanization, we mean making a computer automatically verify that a mathematical proof is correct. Although one might initially think that this is just a matter of having a sufficiently smart proof checker, on par with what mathematicians routinely do, it has appeared that the problem is much more deep, leading logicians to consider alternative foundations of mathematics, more amenable to mechanization: *dependent type theory* [1], [2] (later refered to as DTT). This path has lead to revolutionary ideas, unveiling a deep connection between logic and homotopy theory [3], [4], [5], sprouting into Homotopy Type Theory [6], which is still actively studied today.

At the same time, Girard discovered what has now become linear logic [7] (later refered to as LL), as if he put the most fundamental logical connectives that were known at the time in a particule accelerator, making them collide, break, and reveal that they are, in fact, composed of "smaller", more primitive, ones. This very fundamental discovery has, surprisingly, managed to reconcile two worlds that were thought to be completely disjoint: classical logic, and intuitionistic logic. This is crucial, as modern mathematics are built in a classical framework, yet type theories are intuitionistic in nature. Linear logic has since become a very rich research area, relating logic with many other fields.

1.2. Research problem

Despite their respective importance in the field, and several attempts [8], [9], no satisfactory linear dependent type system has been found yet. Achieving this would be a key milestone, as it would, among other things, improve our ability to do classical reasonings in a dependent setting. The issue is that linear logic expresses finely the dependence of certain formulas with respect to other formulas, seen as resources.

1.3. Our contribution

During the internship, we have studied the problem of defining a linear dependent type theory, by interleaving two complementary viewpoints: syntax and semantics. Indeed, we have

- identified a key type former, the singleton type, as a bridge between LL and DTT
- developed a linear dependent type theory, the Hanaba system. The main technical novelties are
  1. a usage of several contexts to speak of linearity and dependence (modifying contexts with this purpose in mind has been done before, eg [8], but not in the same way)
  2. the usage of the aforementioned singleton type
  3. mixing intrinsic and extrinsic semantics of the λ-calculus, after [10]

  We have not proved standard meta-theoretical results for the system, such as subject reduction, inversion lemmas, canonicity, confluence, strong normalisation, consistency, …
- developed a notion of models for the Hanaba system on top of fibered double categories. A model is, intuitively, a mathematical object in which one can interpret the Hanaba system, but we haven't fully defined the interpretation yet.

- built a “concrete” family of such models. These are useful for several reasons:
  1. they help developping an intuition of what Hanaba models look like
  2. their existence is a key step in proving some meta-theoretical results, such as consistency

  This family of models is more of a framework on top of which to build models, rather that one, specific model, as it can build a model *without semantic constructors* from any fibered double category. Additional assumptions can be made to build models with more features.

1.4. Arguments supporting its validity

The syntactic and semantics of the Hanaba system were developed simultaneously. Where it was not clear which option to choose when defining the type system, the same problem has been studied from the semantical point of view, and reciprocally. Hence, even if we haven't had the time to prove basic properties of the system, which would support its validity, the system itself and its models have emerged naturally, so we are confident that, with enough time, we could show its consistency.

1.5. Summary and future work

First and foremost, the semantics are not on par with the syntax. Indeed, we have not provided the semantic counterparts to universes, with, coproduct and the exponential. We expect their treatement to be similar to that of models of DTT and linear logic, repsectively.

That being said, some obvious, yet important, technicalities have not been covered during the internship, for time reasons (the size of the system rules out proving these theorems “by hand”, and formalizing the whole system would itself probably require months of work): confluence, strong normalization, canonicity, subject reduction, soundness, completeness, the actual interpretation of the calculus in the model, …

Also, we have not “properly” handled universes: we have assumed $\mathsf{Type} : \mathsf{Type}$, as is usually done in prototype theories (again for time reasons). Fixing this should be completely straightforward, as is done with other DTTs, as our extensions do not interfere with universes.

Besides these minor points (in the sense that solving those problems would probably not require new ideas, as these are common problems which have already been solved in the literature for a wide variety of systems), the Hanaba system embeds only the intuitionistic fragment of linear logic [11]. Crucially, it lacks an involutive negation. We believe such an involutive negation could be added, syntactically by supporting continuations, semantically by considering chirality categories.

1.6. Outline of the document

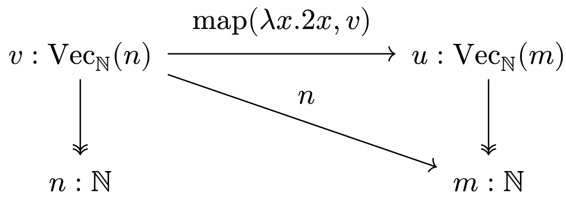
The document starts with Section 2, which presents the main ideas of the syntactic system. Then, it follows up with Section 3, which presents exhaustively the definition of the basis of a Hanaba model. Aquaintance with Grothendieck fibrations is required in order to understand it; in case the reader needs some reminders, they can read Appendix A.

Most of the proofs have been moved to the appendix. Yet, the appendix also contains the construction the the chain models in Appendix F. We suggest reading that section of the appendix, if time permits, above others, as it's the one that contains more novel ideas.

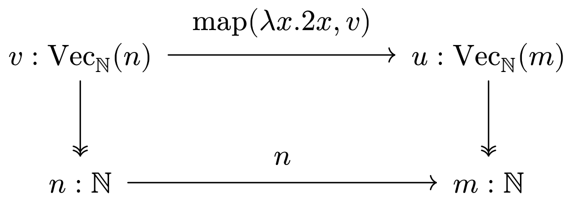
2. Syntax of the Hanaba calculus

Developing a linear dependent type theory therefore involves introducing a more fine-grained control of dependency in a dependent type theory. The main obstruction in doing so is that, in dependent type theory, types themselves can depend (or, at least, observe) the resources whose usage we carefully track. Specifically, it appears that in the current presentation of dependent type theories, substitutions between contexts do not bother with keeping track of what each term of the substitution actually depends on: every term simply implicitely depends on the whole domain context.

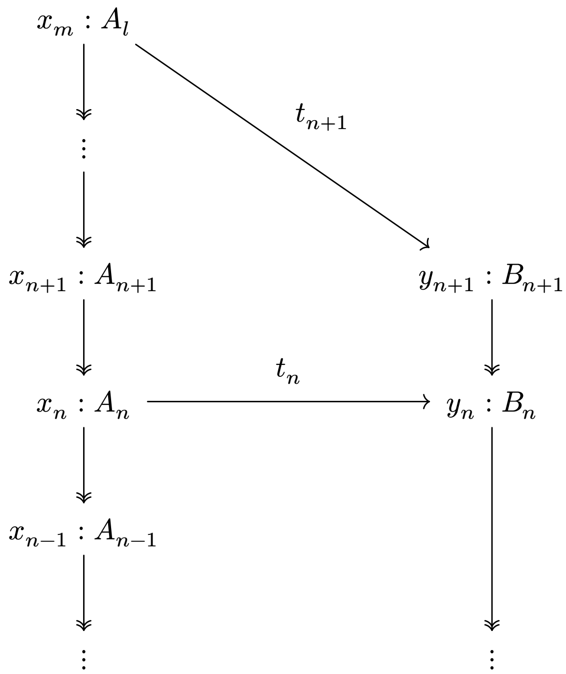
Consider, for instance, a vector type $\mathrm{Vec}_A(n)$ of lists of size n, and of elements of type A. Consider now the substitution that doubles each element in such a list of natural numbers, leaving its size unchanged. In usual DTT, this would look as follows



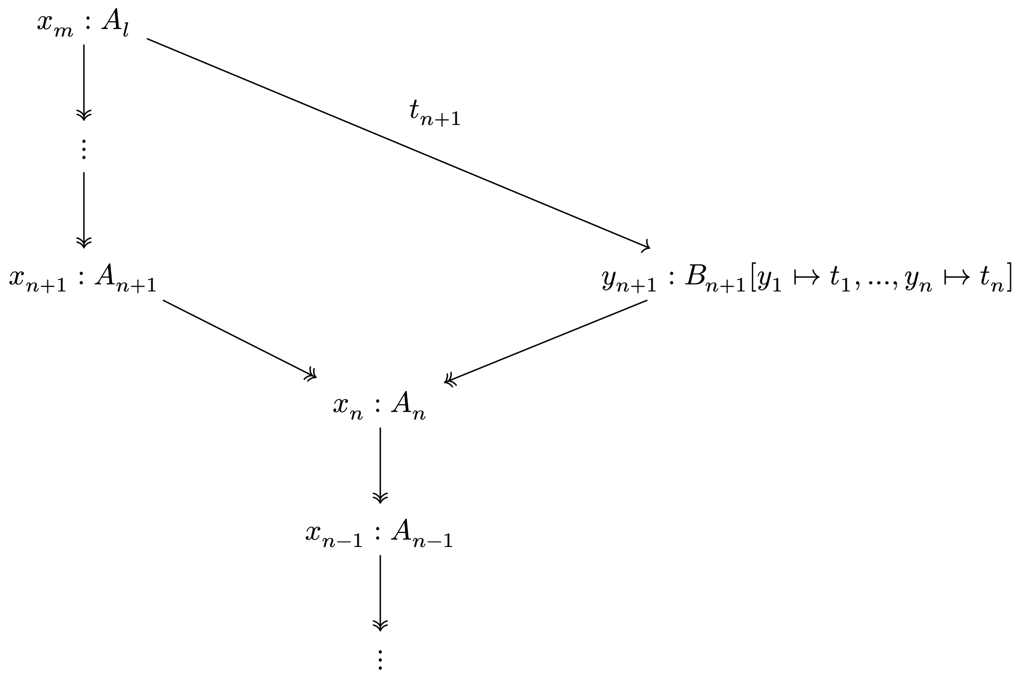
But actually, the term for the variable m is just n, it does not depend on v. So one could be more parsimonious



We can therefore consider a more general form of substitution, where each variable of the codomain context also specifies up to which layer it depends. That is, a substitution might look like so



such term t_{n+1} is exactly the same as in the following situation, where we pull back everything below it



that is, it's a substitution

$$x_{n+1} : A_{n+1}, ..., x_m : A_m \longrightarrow B_{n+1}[y_1 \mapsto t_1, ..., y_n \mapsto t_n]$$

above the context $x_1 : A_1, ..., a_n : A_n$. This naturally suggests that we are interested in judgements of the form

$$\Gamma \vDash \Delta \vdash t : A$$

where Δ proves A over the context Γ.

We furthermore note that context substitutions are not the only place where dependent type theory is "wasteful" with respect to dependency. Indeed, in DTT, the primitive function type is Π types, which make both the resulting value and the resulting type depend on the input. While this should be expressible in a linear dependent setting, we need finer-grained primitive function spaces, expressing that only the resulting value, or the resulting type, depend on the input, because we cannot simply discard a value that we don't care about in a linear setting. The former type former is the linear implication $A \multimap B$, with constructor $\lambda x.t$ where x appears in t but not in B, and the latter is a universal quantification $\forall_{x:A} B$, with constructor $\Lambda x.t$ where x appears in B but not in t.

From this, we deduce two key insights. First, the judgement $\Gamma \vDash \Delta \vdash t : A$ is not enough, because we need to distinguish between variables that can appear in the type, but not in the term, and vice versa. Hence, we introduce a third context Θ in

$$\Gamma \vDash \Theta \mid \Delta \vdash t : A$$

where variables in Θ can only be used in types, and variables in Δ can only appear in t. Hence, the introduction rules for the two type formers that we have described are

$$\frac{\Gamma \vDash \Theta \mid \Delta, x : A \vdash t : B}{\Gamma \vDash \Theta \mid \Delta \vdash \lambda x.t : A \multimap B} \qquad \frac{\Gamma \vDash \Theta, x : A \mid \Delta \vdash t : B}{\Gamma \vDash \Theta \mid \Delta \vdash \Lambda x.t : \forall_{x:A} B}$$

Secondly, this is not enough to express Π types, because variables so far cannot appear both in terms and types. Rather than introducing a third constructor $\lambda x.t$, introducing variables in both contexts at once, we introduce a singleton type $\{a\}_A$ for $a : A$. Elements of $\{a\}_A$ are thought of as those equal to a, but the term a itself appears in the type, whereas its inhabitants appear as terms. Thus, we can decompose

$$\Pi_{a:A} B \coloneqq \forall_{a:A} !\{a\}_A \multimap B$$

Similarly, we can decompose the Σ type in a tensor product type $A \otimes B$ and an existential type $\exists_{a:A} B$, and have a similar equation

$$\Sigma_{a:A} B \coloneqq \exists_{a:A} \{a\}_A \& B$$

Combining these two insights requires some care though. Indeed, in a usual DTT context, one might have the following introduction rule for the singleton type.

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a : \{a\}_A}$$

But in our setting, this is ruled out by the fact that a term cannot, in general, appear both on the left and on the right of the colon in the typing judgement, as these two places live above different contexts, so the following rule is ill-formed:

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash a : A}{\Gamma \vDash \Theta \mid \Delta \vdash a : \{a\}_A}$$

The first step to fix this issue is to relax a little bit the rule to include every term that is definitionally equal to a

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash a' \equiv a : A}{\Gamma \vDash \Theta \mid \Delta \vdash a' : \{a\}_A}$$

Presented as such, if one looks just at the conclusion of this rule, we have solved our issue: two different terms (a' and a) appear on the left and on the right of the colon. Yet, if we look at the premise of the rule, one sees an impossible statement: $a' \equiv a : A$, since they live above different contexts. Hence, we introduce a new typing judgement, stating a heterogeneous definitional equality between a term defined in a term context, and a term defined in a type context

$$\Gamma \vDash \Theta \mid \Delta \vdash u \mapsto v : A$$

with the idea that the introduction rule for the singleton type would be the following

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash a' \mapsto a : A}{\Gamma \vDash \Theta \mid \Delta \vdash a' : \{a\}_A}$$

We will see, however, that in order to make $\mapsto$ a congruence, in order to keep track of variables introduced inside the terms a and a', we need two more contexts in the heterogeneous definitional equality judgement.

The Hanaba system is a intuitionistic linear dependent type system, ie. it is a MLTT [2] with the intuitionistc fragment of LL [11], that is, $\otimes$, $!$, $\multimap$, $\&$ and $\oplus$. Crucially, it lacks negation, and hence $?$ and $⅋$. The full specification of the calculus can be found in Appendix B. We will present here its key ideas.

One of the main points of the Hanaba calculus is the decomposition of Π types as a universal quantification, which gives dependency, a linear implication, which gives linearity, and a singleton type, which relates the two “levels”.

Let us describe the Hanaba calculus step by step. This involves defining the set $\mathbb{T}$ of terms inductively, as well as three kind of contexts:

- Γ, Θ, Σ are "dependent" contexts, that is, contexts used additively, and where each type can depend on previous variables in the context, as is usual in DTT
- Ξ is a "linear" context, that is, it will be used multiplicatively, and types cannot depend on variables in the same context (type can have free variables, though)
- Δ is a "linear" annotated context, that is, same as Ξ, but variables can be annotated with free variables:

$$\Delta ::= \diamond \mid \Delta, \mathbb{V} : \mathbb{T} \mid \Delta, \mathbb{V} \mapsto \mathbb{V} : \mathbb{T}$$

With this in mind, we will define the following judgements

- $\vdash \Gamma$ ctxt: "Γ is a well-formed (dependent) context"
- $\Gamma \vdash \Delta$ ctxt: "Δ is a well-formed (linear) context under (dependend) context Γ"
- $\Gamma \vdash T$ type: "the type T is well-defined in context Γ"
- $\Gamma \vDash \Theta \mid \Delta \vdash t : T$: "$t$ has type T in **global** context Γ, **type** context Θ and **term** context Δ". Note that Γ is an intuitionistic context whose variables can appear both in t and in T (intuitionistic in the sense that variables can appear any number of times, or, equivalently, that it will be used additively in typing rules), Θ is an intuitionistic context whose variables can appear only in T, and Δ is a linear context (its variables must appear exactly once) whose variables can appear only in t.
- $\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto v : A$: "$u$ is under v of type A in global context Γ, type context Θ, exclusive type context Σ, term context Δ" and shared term context Ξ. As before, Γ is an intuitionistic context whose variables can appear in u, v and A, Θ is an intuitionistic context whose variables can appear only in v and A, Σ is an intuitionistic context whose variables can appear only in A, Δ is an enriched linear context whose variables can appear only in u, and Ξ is a linear context whose variables can appear only in u and v
- $\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : A$: "$u$ is definitionally equal to v of type A in global context Γ, type context Θ, term context Δ"
- $\Gamma \vdash A \equiv B$ type: "A is a type definitionally equal to B in global context Γ"

2.1. Definitional equality

We will not give here the exact rules for the definitional equality, but they can be easily intuited: it is an equivalence relation, that is a congruence with respect to the syntax, and which might closed under η (if one wishes the theory to be extensional).

Similarly, the definitional equality between types could be expressed purely in terms of the definitional equality of terms, which happen to be types, but it is technically easier to include it as an additional judgement.

2.2. Contexts

Dependent context formation is very much akin to what happens in usual DTT

$$\frac{}{\vdash \diamond \text{ ctxt}}\text{Ctxt-NL-empty} \qquad \frac{\vdash \Gamma \text{ ctxt} \quad \Gamma \vdash A \text{ type}}{\vdash \Gamma, x : A \text{ ctxt}}\text{Ctxt-NL-cons}$$

Linear context formation, on the other hand, happens under a dependent context

$$\frac{}{\Gamma \vdash \diamond \text{ ctxt}}\text{Ctxt-L-empty} \qquad \frac{\Gamma \vdash \Delta \text{ ctxt} \quad \Gamma \vdash A \text{ type}}{\Gamma \vdash \Delta, x : A \text{ ctxt}}\text{Ctxt-L-cons}$$

$$\frac{\Gamma \vdash \Delta \text{ ctxt} \quad \Gamma \vDash \diamond \mid \diamond \vdash y : A}{\Gamma \vdash \Delta, x \mapsto y : A \text{ ctxt}} \text{Ctxt-L-cons'}$$

2.3. Universe

In order to keep the Hanaba system simple, we assume type-in-type. This is blatantly incoherent, but universes are not the construction under scrutiny here, and replacing it with a universe hierarchy is orthogonal to our work.

$$\mathbb{T} \coloneqq \dots \mid \text{Type}$$

$$\frac{}{\Gamma \vdash \text{Type type}} \text{Type-F} \qquad \frac{\Gamma, \Theta \vdash A \text{ type}}{\Gamma \vDash \Theta \mid \diamond \vdash A : \text{Type}} \text{Type-I}$$

The rule Type-I is particularly important as it shows that a type has access to variables that occur in Γ and Θ, but not in Δ.

2.4. Variables

Assume we have a fixed, infinite set of variables $\mathbb{V}$. A term can be a variable

$$\mathbb{T} \coloneqq \dots \mid \mathbb{V}$$

Judgement for typing variables are the axiom rules

$$\frac{x : A \in \Gamma \quad \vdash \Gamma, \Theta \text{ ctxt}}{\Gamma \vDash \Theta \mid \diamond \vdash x : A} \text{Ax-NL} \qquad \frac{\Gamma, \Theta \vdash x : A \text{ ctxt}}{\Gamma \vDash \Theta \mid x : A \vdash x : A} \text{Ax-L}$$

$$\frac{\Gamma, \Theta \vdash x \mapsto y : A \text{ ctxt}}{\Gamma \vDash \Theta \mid x \mapsto y : A \vdash x : A} \text{Ax-L'}$$

Note that free variables of the term in a judgement of the form $t : A$ can only occur in Γ, any number of times, or in Δ, exactly once.

2.5. Function space

$$\mathbb{T} \coloneqq \dots \mid \lambda \mathbb{V}.\mathbb{T} \mid \mathbb{T}\mathbb{T} \mid \mathbb{T} \multimap \mathbb{T}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \multimap B \text{ type}} \multimap\text{-F} \qquad \frac{\Gamma \vDash \Theta \mid \Delta, x : A \vdash t : B}{\Gamma \vDash \Theta \mid \Delta \vdash \lambda x.t : A \multimap B} \multimap\text{-I}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u : A \multimap B \quad \Gamma \vDash \Theta \mid \Delta_2 \vdash v : A}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash uv : B} \multimap\text{-E}$$

With Γ and Θ fixed, these definitions correspond to the usual linear logic implication.

2.6. Universal quantification

Note that the linear map space, $A \multimap B$, does not bind the argument variable at the type-level: B does not depend on $a : A$. Dually, Hanaba has a universal quantification type, whose terms are those that bind a variable but do not “use” it

$$\mathbb{T} \coloneqq \dots \mid \Lambda \mathbb{V}.\mathbb{T} \mid \mathbb{T}@\mathbb{T} \mid \forall_{\mathbb{V}:\mathbb{T}}\mathbb{T}$$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, a : A \vdash B \text{ type}}{\Gamma \vdash \forall_{a:A} B \text{ type}} \forall\text{-F} \qquad \frac{\Gamma \vDash \Theta, x : A \mid \Delta \vdash t : B \quad \Gamma, \Theta \vDash \Delta \text{ ctxt}}{\Gamma \vDash \Theta \mid \Delta \vdash \Lambda x.t : \forall_{x:A} B} \forall\text{-I}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash u : \forall_{a:A} B \quad \Gamma, \Theta \vDash \diamond \mid \diamond \vdash v : A}{\Gamma \vDash \Theta \mid \Delta \vdash u@v : B[a \mapsto v]} \forall\text{-E}$$

Note that, in the rule ∀-I, the bound variable is not added in the term context Δ, but in the type context Θ, which has not axiom rules. Variables in Θ can only be accessed when Θ is merged with the global context Γ, as in rule ∀-E. Dually, the latter rule does not require the argument to be defined in the same context as the application itself.

2.7. Singleton type

The two previous sections have introduced, respectively, linearity and dependency, in a separate fashion: you can either bind a variable on which a type can depend, or you can bind it so that the term (linearly) depends on it. This is not enough to express, say, Π types, where the variable bound is used both in the term and in the type. To link these two worlds, we need the singleton type $\{a\}_A$. Elements of $\{a\}_A$, where $a : A$, are thought of as being definitionally equal to a. However, inhabitants of $\{a\}_A$ live at the term level, whereas a itself lives at the type level. To express a definitional equality between terms that live at different levels, we need a new judgement: $u \mapsto a : A$. The idea is that having a $u : \{a\}_A$ is the same as $u \mapsto a : A$ holding, and $\mapsto$ should additionally be a congruence.

$$\frac{\Gamma \vDash \diamond \mid \diamond \vdash a : A}{\Gamma \vdash \{a\}_A \text{ type}}\,\{\}\text{-F} \qquad \frac{\Gamma \vDash \Theta \mid \diamond \mid \Delta \mid \diamond \vdash t \mapsto a : A}{\Gamma \vDash \Theta \mid \Delta \vdash \text{loop } t : \{a\}_A}\,\{\}\text{-I}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_2, x \mapsto a : A \vdash v : B \qquad \Gamma \vDash \Theta \mid \Delta_1 \vdash u : \{a\}_A}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let loop } x = u \text{ in } v : B}\,\{\}\text{-E}$$

The rules for the judgement $\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto v : A$ follow closely the typing rules. For instance, we have the following rules

$$\frac{\Gamma, \Theta, \Sigma \vdash x \mapsto u : A \text{ ctxt}}{\Gamma \vDash \Theta \mid \Sigma \mid x \mapsto u : A \mid \diamond \vdash x \mapsto u : A}\,\text{Ax-}{\mapsto}\text{-L'} \qquad \frac{\Gamma, \Theta, \Sigma \vdash x : A \text{ ctxt}}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid x : A \vdash x \mapsto x : A}\,\text{Ax-}{\mapsto}\text{-L}$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi, x : A \vdash u \mapsto v : B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \lambda x.u \mapsto \lambda x.v : A \multimap B}\,{\multimap}\text{-I-}{\mapsto} \qquad \frac{\Gamma \vDash \Theta \mid \Sigma, x : A \mid \Delta \mid \Xi \vdash u \mapsto v : B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \Lambda x.u \mapsto \lambda x.v : \forall_{x:A} B}\,\forall\text{-I-}{\mapsto}$$

with this, we can reconstruct the intuitionistic Π type as

$$\Pi_{a:A} B \coloneqq \forall_{a:A} !\{a\} \multimap B$$

3. Semantics of the Hanaba calculus

Following the logic of [1], [12], we will define a semantic couterpart to the previously defined system. Our definition will rely on fibered double category, which is inspired by comprehension categories [13] and fibrational models of DTT. It will also rely on extrinsic term semantics, as in [10].

More specifically, we will define the notion of *fibered double category*, which is a framework in which to speak of models of DTT, as will be explained later. Similarly, we will use monoidal closed categories as the backbone of models of linear logic. As syntactically our system has three contexts, stacked onto each other, semantically, our models will be defined as two fibered double category sitting on top of each other (representing respectively the contexts Γ and the context Θ), as well monoidal closed categories on top of it, representing the category whose elements are contexts Δ.

In a way, the separation between the type context, and the term context is reminescent of some restrictions of proof irrilevant universes. Indeed, terms well typed in our system are those where variables of the type context are only used to build types, which can be thought of as everything that is erased at compile time. Under this interpretation, a term is well-typed if, after performing a "type erasure", it doesn't contain any variable of the type context anymore.

This is not yet a fully semantical condition, though. A better way to phrase this would be to say that the behavior of the term (what is obtained after the erasure) is invariant under substitutions that only affect variables that appear in the type context.

To express this semantically, one has to consider a category of type contexts, on top of which is a fibered model of linear logic. For each type context Θ, there is a fiber in which morphisms $\Delta \to A$ represent terms t such that $\Theta \mid \Delta \vdash t : A$. However, such a term can still "depend" on Θ, so we additionally have a model $\mathcal{M}$ of untyped lambda calculus in which to interpret t: $\{\!|t|\!\} \in \mathcal{M}$ in such a way that for any substitution $\rho : \Theta' \to \Theta$, we have $\{\!|t\{\rho\}|\!\} = \{\!|t|\!\}$.

Of course, this all actually lives on top of a global context Γ, that is, in a fiber above Γ. Hence a Hanaba model is two fibrations, one sitting above the other, with a fibered model of untyped lambda calculus.

3.1. Models of untyped linear λ calculus

Hanaba models are partly built as extrinsic models of λ calculus, in the spirit of [10]. The motivating example is the following:

$$\diamond \vDash \alpha : \mathrm{Type} \mid \diamond \vdash \iota_2 @\alpha @1 * : \alpha \oplus 1$$

Note that, in this term, the variable α appears, but is irrilevant as to the "meaning" of the term. That is, if we substitute α for $\mathbb{N}$ or 1, it doesn't change its computational behavior, yet the two terms are not exactly equal. Hence, we would like to have a "type erasure" operation that erases from the term the parts of it that are not relevant for its computational behavior, turning $\iota_2 @\alpha @1 *$ into $\iota_2 *$. Of course, the resulting term is not type correct, as the type information has been forgotten, so such a "type erasure" operation must have as codomain an untyped setting.

Semantically, such codomain will be a fibration over a base category whose fibers are models of untyped linear λ calculus. Let us first define the latter, as in [14].

Definition 3.1.1 (Model of linear λ calculus)

A *model of linear λ calculus* is a monoidal closed category $(\mathcal{C}, I, \otimes, \multimap)$ with a *universe* $U : \mathcal{C}$ equipped with a retraction

$$U \multimap U \underset{\mathrm{app}}{\overset{\mathrm{lam}}{\rightleftarrows}} U$$

ie. such that

$$\mathrm{app} \circ \mathrm{lam} = \mathrm{id}_{U \multimap U}$$

♣

Definition 3.1.2 (Morphism of model of linear λ calculus)

A morphism between a model $(C, I, \otimes, \multimap, U, \mathrm{app}, \mathrm{lam})$ and $(C', I', \otimes', \multimap', U', \mathrm{app}', \mathrm{lam}')$ is a closed monoidal functor $F : C \to C'$ such that

$$F(U) = U'$$
$$F(\mathrm{app}) = \mathrm{app}'$$
$$F(\mathrm{lam}) = \mathrm{lam}'$$

♣

Models of linear λ calculus and their morphisms form a category **LλMod** , which is a subcategory of **Cat**.

Note that **LλMod** is exactly the category of diagrams of shape $\mathcal{S}$ in the category of monoidal closed categories, where $\mathcal{S}$ is the free monoidal closed category with an object P, as well as a retraction $P \multimap P \lhd P$.

Definition 3.1.3 (LλMod-fibered fibration)

A **LλMod**-fibered fibration is a fibration $p : \mathcal{C} \to \mathcal{D}$ such that the fiber pseudo-functor p^{-1} factors through the forgetful functor **LλMod** $\to$ **Cat**.

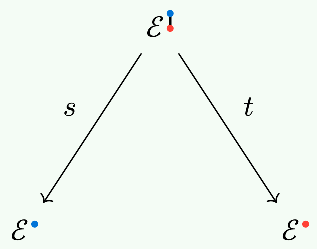
♣

3.2. Fibered double categories

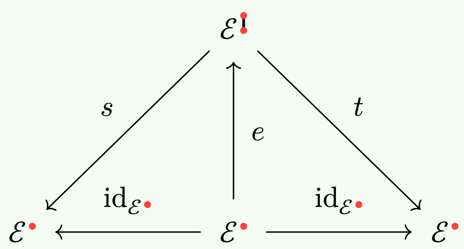
Hanaba models are inspired by the fibrational approach of categorical semantics [15], double categories [16]. Hence, let us define a notion of *fibered double category*, mixing the two notions.

Definition 3.2.1 (Double category)

A *double category* is a pair of categories $\mathcal{E}^{\bullet}$ and $\mathcal{E}^{\updownarrow}$ forming an internal graph in the category **Cat**



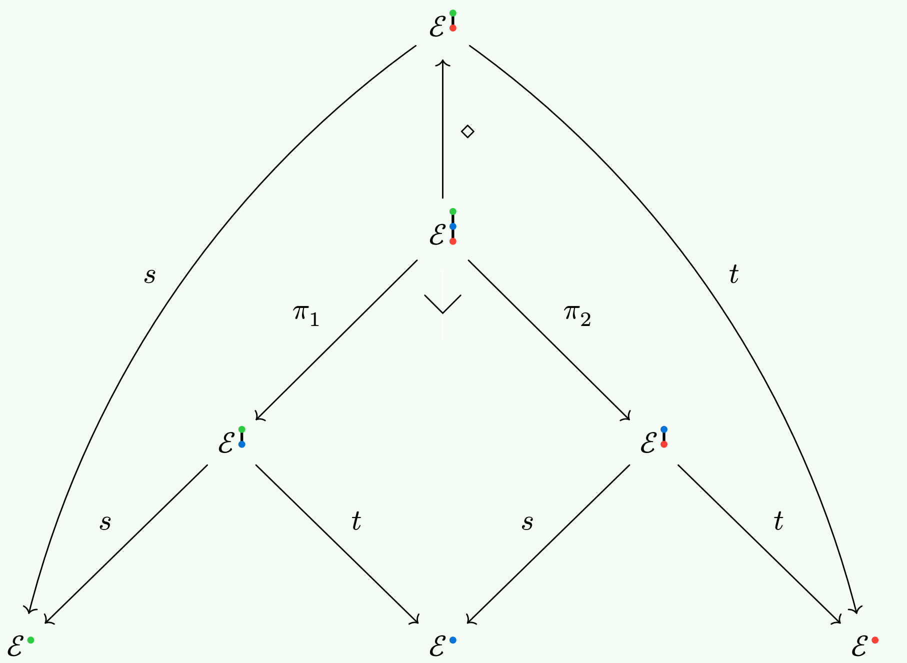
where s is the *source functor* and t the *target functor*, equipped with a *unit functor* $e : \mathcal{E}^{\bullet} \to \mathcal{E}^{\updownarrow}$ making the following diagram commute



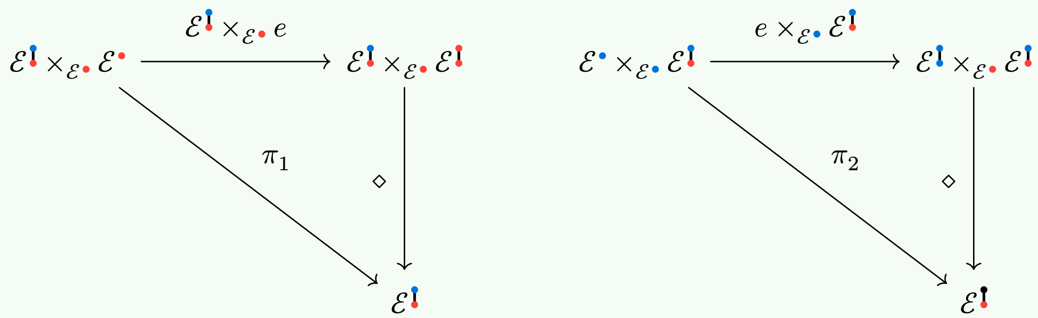
and a *composition functor*

$$\diamond : \mathcal{E}^{\updownarrow\updownarrow} \to \mathcal{E}^{\updownarrow}$$

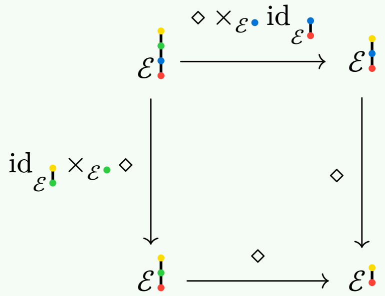
making the following diagram commute



The unit must behave as an identity with respect to composition, that is, the following two triangles must commute



Furthermore, the composition must be associative, that is, the following diagram must commute (note that this diagram is well-defined by definition of $\diamond$):



If we are given a double category $\left(\mathcal{E}^{\bullet}, \mathcal{E}^{\vdots}, s, t, e, \diamond\right)$, we will denote by a horizontal arrow a morphism in $\mathcal{E}^{\bullet}$, and by a vertical arrow an object of $\mathcal{E}^{\vdots}$:

$$\begin{array}{c} A \\ \alpha \Big\downarrow \\ B \end{array}$$

where $s(\alpha) = A$ and $t(\alpha) = B$. Furthermore, we will say that a square as follows commutes

$$\begin{array}{ccccc}
A_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_n} & B_1 \\
\alpha_1 \Big\downarrow & & & & \Big\downarrow \beta_1 \\
\vdots & & & & \vdots \\
\alpha_k \Big\downarrow & & & & \Big\downarrow \beta_m \\
A_{k+1} & \xrightarrow{g_1} & \cdots & \xrightarrow{g_r} & B_{m+1}
\end{array}$$

if there exists a morphism

$$\varphi : \alpha_k \diamond \cdots \diamond \alpha_1 \longrightarrow \beta_m \diamond \cdots \diamond \beta_1$$

such that

$$s(\varphi) = f_n \circ \cdots \circ f_1$$
$$t(\varphi) = g_r \circ \cdots \circ g_1$$

Definition 3.2.2 (Fibered double category)

A double category $\left(\mathcal{E}^{\bullet}, \mathcal{E}^{\text{¡}}, s, t, e, \diamond\right)$ is said to be *fibered* if t is a split cloven Grothendieck fibration, the unit is a fibration morphism

$$e : \mathrm{id}_{\mathcal{E}^{\bullet}} \longrightarrow t$$

and so is the composition

$$\diamond : t \circ \pi_2 \longrightarrow t$$

♣

Intuitively, in a fibered double category $\left(\mathcal{E}^{\bullet}, \mathcal{E}^{\text{¡}}, s, t, e, \diamond\right)$, $\mathcal{E}^{\bullet}$ represents the category of contexts, with morphism substitutions, and $\mathcal{E}^{\text{¡}}$ represents the category of types on top of a context. The functor t is the forgetful functor that, to each type associates the context on top of which it is defined. s, on the other hand, is the context extension: if we have a type A on top of a context Γ, $s(A)$ is the context $\Gamma.A$. The unit of the double category is the functor that, to each context associates the unit type on top of that context, and $\diamond$ is the dependent sum.

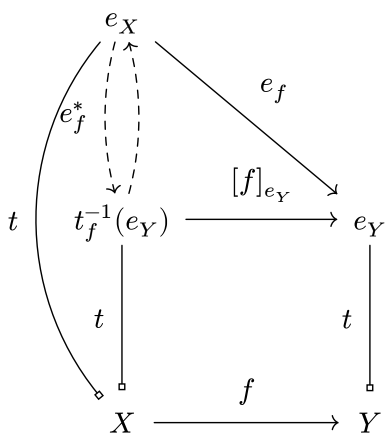
t being a fibration makes the notion of a *category of types above a context* Γ meaningful, because we can consider the fiber t_Γ^{-1}. Furthermore, for each substitution $\rho : \Delta \to \Gamma$, it can consider its action $t_\rho^{-1} : t_\Gamma^{-1} \to t_\Delta^{-1}$ which actually *performs* the substitution on types and terms. Finally, the fact that e and $\diamond$ are fibration morphisms is to ensure the substitution of the unit type above on fiber gives the unit type above the other fiber, and similarly that the substitution of the dependent sum is the dependent sum of the substitution. This is exactly what the following propositions will show:

Proposition 3.2.3

Let $f : X \to Y$ be a morphism in $\mathcal{E}^\bullet$. We have a canonical isomorphism

$$e_f^* : e_X \cong t_f^{-1}(e_Y)$$

Proof. f is a cartesian morphism for the identity fibration, hence e_f is itself cartesian for t. Hence, there exist e_f^* and its inverse making the following diagram commute



□

Definition 3.2.4 (Morphism extension)

Given a morphism $f : X \to Y$ in $\mathcal{E}^\bullet$ and $R : t_Y^{-1}$ be above Y, the *extension of f by R* is

$$f^R := s([f]_R)$$

Proposition 3.2.5

Given a morphism $f : X \to Y$ in $\mathcal{E}^\bullet$, and $R : t_Y^{-1}$ and $S : t_{s(R)}^{-1}$, the morphism $\left([f]_R^t, [f^R]_S^t\right)$ is cartesian for $t \circ \pi_2$.

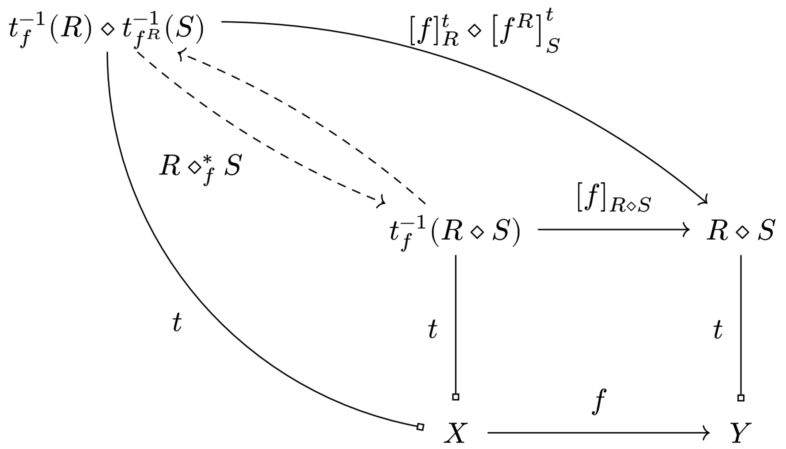
Proof. See Appendix G. □

Proposition 3.2.6

Given a morphism $f : X \to Y$ in $\mathcal{E}^\bullet$, and $R : t_Y^{-1}$ and $S : t_{s(R)}^{-1}$, we have a canonical isomorphism

$$R \diamond_f^* S : t_f^{-1}(R) \diamond t_{f^R}^{-1}(S) \cong t_f^{-1}(R \diamond S)$$

Proof. Because $\left([f]_R^t, [f^R]_S^t\right)$ is cartesian, and because $\diamond$ is a fibration morphism, $[f]_R^t \diamond [f^R]_S^t$ is cartesian. Hence, there exist unique maps making the following commute

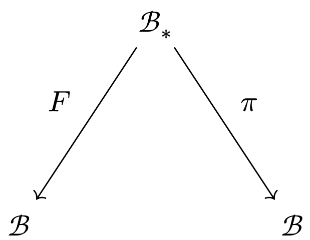


□

3.3. Foundations of Hanaba models

We will now proceed to defining a Hanaba model. This definition will span over the next few sections, as we will enumerate all the data in this definition. A summary of this enumeration can be found in Appendix E.

Suppose we have a fibered double category (1)



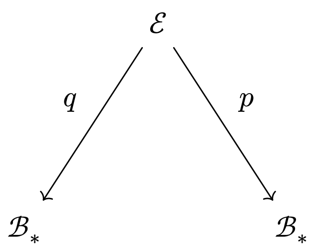
with unit (2)

$$\langle\rangle : \mathcal{B} \to \mathcal{B}_*$$

and with multiplication (3)

$$-,- : \mathcal{B}_* \times_{\mathcal{B}} \mathcal{B}_* \longrightarrow \mathcal{B}_*$$

as well as an other fibered double category (4)



with unit (5)

$$1 : \mathcal{B}_* \to \mathcal{E}$$

and multiplication (6)

$$\Sigma : \mathcal{E} \times_{\mathcal{B}_*} \mathcal{E} \longrightarrow \mathcal{E}$$

Suppose we also have a **LλMod**-fibered fibration (7) $\upsilon : \mathcal{C} \to \mathcal{B}$. For $\Gamma : \mathcal{B}$, let us note $\mathcal{C}_\Gamma := \upsilon_\Gamma^{-1}$ and U_Γ the universe of $\mathcal{C}_\Gamma$.

Let's call $\mathcal{B}$ the *global context category*, as its elements will be global contexts (ie. contexts that we called Γ in the syntactic part). $\mathcal{B}_*$ is the *pointed context category*, ie. contexts that we called Θ in the syntactic part. The name comes from the fact that, in the chain model, $\mathcal{B}_*$ will be constructed as the category of pointed global contexts. Finally, let's call $\mathcal{E}$ the *type category*.

3.4. Global contexts $\mathcal{B}$

For $\Gamma : \mathcal{B}$, let us note $\mathrm{Ctxt}(\Gamma)$ the fiber π_Γ^{-1} above Γ. For a context $\Theta : \mathrm{Ctxt}(\Gamma)$, and a morphism $\rho : \Gamma' \to \Gamma$, let

$$\Theta\{\rho\} := \pi_\rho^{-1}(\Theta)$$

This defines a function

$$-\{\rho\} : \mathrm{Ctxt}(\Gamma) \to \mathrm{Ctxt}(\Gamma')$$

which we call *substitution action along ρ*.

For $\Theta : \mathrm{Ctxt}(\Gamma)$, let us note

$$\Gamma, \Theta := F(\Theta)$$

We have purpusefully chosen this notation to be the same as the multiplication $-, -$, since we have

$$\Gamma, (\Theta_1, \Theta_2) = (\Gamma, \Theta_1), \Theta_2$$

which we can therefore denote as

$$\Gamma, \Theta_1, \Theta_2$$

3.4.1. Global context projection

Suppose there is a natural transformation $\mathbf{p} : F \Rightarrow \pi$ (8) such that, for every context $\Gamma : \mathcal{B}$, we have

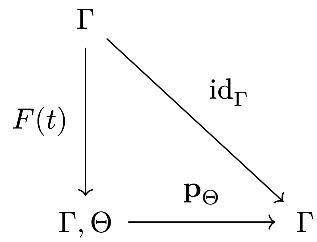
$$\mathbf{p}_{\langle\rangle_\Gamma} = \mathrm{id}_\Gamma$$

3.4.2. Global terms

For a context $\Gamma : \mathcal{B}$, and a pointed context $\Theta : \mathrm{Ctxt}(\Gamma)$, a *global term at* Θ is a morphism $t : \langle\rangle_\Gamma \to \Theta$ in $\mathrm{Ctxt}(\Gamma)$. We write $\mathrm{Tm}(\Gamma \vDash \Theta)$ for the set of global terms at Θ. Note that a global term $t : \mathrm{Tm}(\Gamma \vDash \Theta)$ induces a section of the global context projection. Indeed, we have

$$\begin{array}{ccc} \Gamma, \langle\rangle_\Gamma & \xrightarrow{\mathbf{p}_{\langle\rangle_\Gamma}} & \Gamma \\ {\scriptstyle F(t)}\downarrow & & \downarrow{\scriptstyle \pi(t)} \\ \Gamma, \Theta & \xrightarrow{\mathbf{p}_\Theta} & \Gamma \end{array}$$

Since $\pi(t) = \mathrm{id}_\Gamma$, $\Gamma, \langle\rangle_\Gamma = \Gamma$ and $\mathbf{p}_{\langle\rangle_\Gamma} = \mathrm{id}_\Gamma$, we have that the following diagram commutes



Let $\rho : \Gamma' \to \Gamma$ be a substitution, and $t : \mathrm{Tm}(\Gamma \vDash \Theta)$. Let us note $t\{\rho\} := \pi_\rho^{-1}(t) \circ \langle\rangle_\rho^* : \langle\rangle_\Delta \to \Theta\{\rho\}$ in $\mathrm{Ctxt}(\Gamma')$. This defines the *substitution action along* ρ:

$$-\{\rho\} : \mathrm{Tm}(\Gamma \vDash \Theta) \to \mathrm{Tm}(\Gamma' \vDash \Theta\{\rho\})$$

For $\Theta : \mathrm{Ctxt}(\Gamma)$, suppose there is a term $\mathbf{v}_\Theta : \mathrm{Tm}(\Gamma, \Theta \vDash \Theta\{\mathbf{p}_\Theta\})$ (9) .

3.4.3. Global context extension

Let global contexts Γ_1 and Γ_2, and $\Theta : \mathrm{Ctxt}(\Gamma_2)$ be a pointed context. For any substitution $\rho : \Gamma_1 \to \Gamma_2$ and a term $t : \mathrm{Tm}(\Gamma_1 \vDash \Theta\{\rho\})$, there exists a substitution $\langle \rho, t\rangle : \Gamma_1 \to \Gamma_2, \Theta$ (10) such that

$$\begin{aligned}
\mathbf{p}_\Theta \circ \langle \rho, t\rangle &= \rho \\
\mathbf{v}_\Theta\{\langle \rho, t\rangle\} &= t \\
\langle \rho, t\rangle \circ \rho' &= \langle \rho \circ \rho', t\{\rho'\}\rangle \\
\langle \mathrm{id}_\Gamma, \mathbf{v}_\Theta\rangle &= \mathrm{id}_{\Gamma,\Theta}
\end{aligned}$$

3.5. Pointed contexts $\mathcal{B}_*$

For a global context $\Gamma : \mathcal{B}$, a pointed context $\Theta : \mathrm{Ctxt}(\Gamma)$, let us note $\mathrm{Ty}(\Gamma \vDash \Theta) = p_\Theta^{-1}$ the category of *types* above Γ and Θ. We will sometimes omit the Γ, and simply note it $\mathrm{Ty}(\Theta)$. For a type $A : \mathrm{Ty}(\Gamma \vDash \Theta)$, let us write

$$\Theta.A := q(A)$$

For a pointed substitution $\rho : \Theta' \to \Theta$ in $\mathrm{Ctxt}(\Gamma)$, and a type $A : \mathrm{Ty}(\Gamma \vDash \Theta)$, let us define

$$A\{\rho\} := p_\rho^{-1}(A)$$

This defines a map

$$-\{\rho\} : \mathrm{Ty}(\Gamma \vDash \Theta) \longrightarrow \mathrm{Ty}(\Gamma \vDash \Theta')$$

called the *action along the substitution* ρ.

Suppose there is a natural transformation $\overline{\mathbf{p}} : q \Rightarrow p$ (11) such that, for every pointed context $\Theta : \mathcal{B}_*$, we have

$$\overline{\mathbf{p}}_{1_\Theta} = \mathrm{id}_\Theta$$

3.5.1. Pointed terms

For a context $\Gamma : \mathcal{B}$, a pointed context $\Theta : \mathrm{Ctxt}(\Gamma)$, and a type $A : \mathrm{Ty}(\Gamma \vDash \Theta)$, a *pointed term of type* A is a morphism $t : 1_\Theta \to A$ in $\mathrm{Ty}(\Gamma \vDash \Theta)$. We write $\mathrm{Tm}(\Gamma \vDash \Theta \vdash A)$ for the set of such terms.

Let $\rho : \Theta_1 \to \Theta_2$ be a pointed substitution in $\mathrm{Ctxt}(\Gamma)$, and $t : \mathrm{Tm}(\Gamma \vDash \Theta_2 \vdash A)$ be a term. Let us note

$$t\{\rho\} := p_\rho^{-1}(t) \circ 1_\rho^* : 1_{\Theta_1} \to A\{\rho\}$$

This defines the *action along the substitution* ρ:

$$-\{\rho\} : \mathrm{Tm}(\Gamma \vDash \Theta_2 \vdash A) \longrightarrow \mathrm{Tm}(\Gamma \vDash \Theta_1 \vdash A\{\rho\})$$

For Γ a global context, $\Theta : \mathrm{Ctxt}(\Gamma)$ and $A : \mathrm{Ty}(\Gamma \vDash \Theta)$, suppose there is a term $\mathbf{v}_A : \mathrm{Tm}(\Gamma \vDash \Theta \vdash A)$ (12) .

Note that a pointed term $t : \mathrm{Tm}(\Gamma \vDash \Theta \vdash A)$ induces a section of the pointed context projection. TODO

3.5.2. Pointed context extension

Let Γ be a global context, $\Theta_1, \Theta_2 : \mathrm{Ctxt}(\Gamma)$ be pointed contexts, and $A : \mathrm{Ty}(\Gamma \vDash \Theta_2)$ be a type. For any pointed substitution $\rho : \Theta_1 \to \Theta_2$, and any term $t : \mathrm{Tm}(\Gamma \vDash \Theta_1 \vdash A\{\rho\})$, there exists a substitution $\langle \rho, t \rangle_\Gamma : \Theta_1 \to \Theta_2.A$ (13) such that

$$\begin{aligned} \overline{\mathbf{p}}_A \circ \langle \rho, t \rangle_\Gamma &= \rho \\ \mathbf{v}_A\{\langle \rho, t \rangle_\Gamma\} &= t \\ \langle \rho, t \rangle_\Gamma \circ \rho' &= \langle \rho \circ \rho', t\{\rho'\} \rangle_\Gamma \\ \langle \mathrm{id}_\Theta, \mathbf{v}_A \rangle_\Gamma &= \mathrm{id}_{\Theta.A} \end{aligned}$$

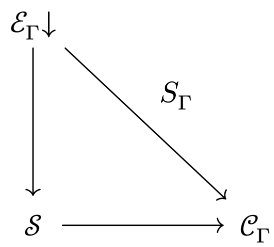
3.5.3. Linear structure

Assume that p is a **MonCat**-fibered fibration, where **MonCat** is the category of symmetric monoidal closed categories and monoidal closed functors. Suppose, furthermore, that the unit of each fiber above Θ is exactly 1_Θ, and that it is a final object in the fiber category.

Let us note $\mathcal{E}_\Gamma$ the fiber of $\pi \circ p$ above Γ.

Consider the category $\mathcal{E}_\Gamma\!\downarrow$ whose objects are those of $\mathcal{E}$, and morphisms are $\rho : A\{\rho\} \to A$ for $\rho : \Theta \to \Theta'$ and $A : \mathrm{Ty}(\Theta')$.

Suppose there is a *shape functor* $S_\Gamma : \mathcal{E}_\Gamma\!\downarrow \to \mathcal{C}_\Gamma$ (14) that factors through $\mathcal{S}$:



such that, for any substitution ρ, and any types A and B, we have

$$S_\Gamma(\rho : A\{\rho\} \to A) \otimes S_\Gamma(\rho : B\{\rho\} \to B) = S_\Gamma(\rho : (A \otimes B)\{\rho\} \to A \otimes B)$$

Furthermore, for any global context Γ, pointed context $\Theta : \mathrm{Ctxt}(\Gamma)$, suppose there is a monoidal closed functor $\{\!\!\{-\}\!\!\}_{\Gamma \vDash \Theta} : \mathrm{Ty}(\Gamma \vDash \Theta) \to \mathcal{C}_\Gamma$ (15) such that, for every $A : \mathrm{Ty}(\Gamma \vDash \Theta)$,

$$\{\!\!\{A\}\!\!\}_{\Gamma \vDash \Theta} = S_\Gamma(A)$$

Definition 3.5.1 (Linear terms)

Given two types $\Delta, A : \mathrm{Ty}(\Gamma \vDash \Theta)$, a *linear term* is a morphism $t : \Delta \to A$ such that for every pointed substitution $\rho : \Theta' \to \Theta$ in $\mathrm{Ctxt}(\Gamma)$, the following diagram commutes

$$\begin{array}{ccc} \{\!\!\{\Delta\{\rho\}\}\!\!\} & \xrightarrow{\{\!\!\{t\{\rho\}\}\!\!\}} & \{\!\!\{A\{\rho\}\}\!\!\} \\ {\scriptstyle S_\Gamma(\rho)}\downarrow & & \downarrow{\scriptstyle S_\Gamma(\rho)} \\ \{\!\!\{\Delta\}\!\!\} & \xrightarrow{\{\!\!\{t\}\!\!\}} & \{\!\!\{A\}\!\!\} \end{array}$$

Let us note $\mathrm{Ty}_l(\Gamma \vDash \Theta)$ the subcategory of $\mathrm{Ty}(\Gamma \vDash \Theta)$ where morphisms are linear terms. ♣

Lemma 3.5.2

$\mathrm{Ty}_l(\Gamma \vDash \Theta)$ is a sub-monoidal category of $\mathrm{Ty}(\Gamma \vDash \Theta)$. ♡

Proof. As $\mathrm{Ty}_l(\Gamma \vDash \Theta)$ contains every object of $\mathrm{Ty}(\Gamma \vDash \Theta)$, we just need to check that, given two linear terms $t_i : \Delta_i \to A_i$ $(i \in \{1,2\})$, their product $t_1 \otimes t_2$ is still linear. Consider a pointed substitution $\rho : \Theta' \to \Theta$. Note that, because p is a **MonCat**-fibered fibration, p_ρ^{-1} is a monoidal functor, hence

$$(t_1 \otimes t_2)\{\rho\} = t_1\{\rho\} \otimes t_2\{\rho\}$$

and $\{\!\!\{-\}\!\!\}$ is also a monoidal functor, so

$$\{\!\!\{(t_1 \otimes t_2)\{\rho\}\}\!\!\} = \{\!\!\{t_1\{\rho\}\}\!\!\} \otimes \{\!\!\{t_2\{\rho\}\}\!\!\}$$
$$\{\!\!\{t_1 \otimes t_2\}\!\!\} = \{\!\!\{t_1\}\!\!\} \otimes \{\!\!\{t_2\}\!\!\}$$

hence the following diagram commutes

$$\begin{array}{ccc} \{\!\!\{\Delta_1\{\rho\}\}\!\!\} \otimes \{\!\!\{\Delta_2\{\rho\}\}\!\!\} & \xrightarrow{\{\!\!\{t_1\{\rho\}\}\!\!\} \otimes \{\!\!\{t_2\{\rho\}\}\!\!\}} & \{\!\!\{A_1\{\rho\}\}\!\!\} \otimes \{\!\!\{A_2\{\rho\}\}\!\!\} \\ {\scriptstyle S_\Gamma(\rho) \otimes S_\Gamma(\rho)}\downarrow & & \downarrow{\scriptstyle S_\Gamma(\rho) \otimes S_\Gamma(\rho)} \\ \{\!\!\{\Delta_1\}\!\!\} \otimes \{\!\!\{\Delta_2\}\!\!\} & \xrightarrow{\{\!\!\{t_1\}\!\!\} \otimes \{\!\!\{t_2\}\!\!\}} & \{\!\!\{A_1\}\!\!\} \otimes \{\!\!\{A_2\}\!\!\} \end{array}$$

□

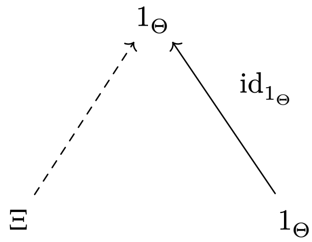
Definition 3.5.3 (Enriched linear context)

An *enriched linear context* over contexts Γ and Θ is a pair of types $\Delta, \overline{\Delta} : \mathrm{Ty}(\Gamma \vDash \Theta)$ with a epimorphism $\Delta \to \overline{\Delta}$ and a term $\delta : \mathrm{Tm}\big(\Gamma \vDash \Theta \vdash \overline{\Delta}\big)$. ♣

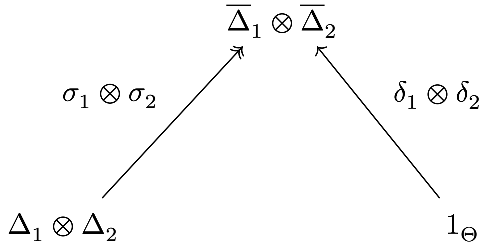
Definition 3.5.4

For a global context $\Gamma : \mathcal{B}$, a pointed context $\Theta : \mathrm{Ctxt}(\Gamma)$, an enriched context $\Delta : \mathrm{Ty}(\Gamma \vDash \Theta)$ and a type $A : \mathrm{Ty}(\Gamma \vDash \Theta)$, let us write $\mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash A)$ for the set of linear terms $\Delta \to A$. ♣

Let $\Xi : \mathrm{Ty}(\Gamma \vDash \Theta)$ be a context. There exists a unique map $\Xi \to 1_\Theta$. Suppose it is an epimorphism. Thus, every context can be seen as an enriched context:

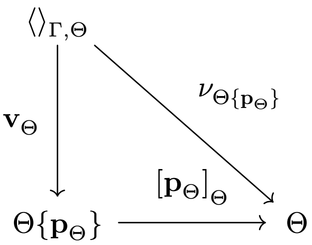


Furthermore, if we have two enriched contexts $\left(\Delta_1, \overline{\Delta}_1, \sigma_1 : \Delta_1 \to \overline{\Delta}_1, \delta_1 : 1_\Theta \to \overline{\Delta}_1\right)$ and $\left(\Delta_2, \overline{\Delta}_2, \sigma_2 : \Delta_2 \to \overline{\Delta}_2, \delta_2 : 2_\Theta \to \overline{\Delta}_2\right)$ over Θ, we can form their tensor product



For this definition to be correct, we need to suppose additionally that the tensor product of epimorphisms is an epimorphism.

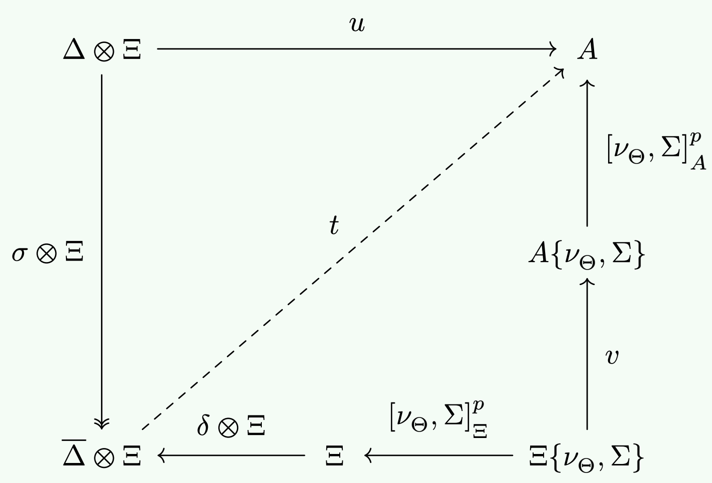
For any $\Theta : \mathcal{B}_*$ above a Γ, let us note $\nu_\Theta : \langle\rangle_{\Gamma,\Theta} \to \Theta$ the following composition



We assume ν_Θ to be an isomorphism. Indeed, in the model, we distinguish between types which are otherwise identical syntactically, that is, if we have a context Γ, Θ and a type A above it (that is, above $\langle\rangle_{\Gamma,\Theta}$), syntactically, it is the same as having a type above Θ, but semantically, they live in different fibers. Hence, we need identify these copies in all the fibers.

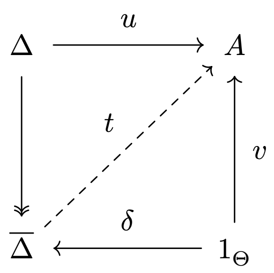
Definition 3.5.5

Given a global context $\Gamma : \mathcal{B}$, a pointed context $\Theta : \mathrm{Ctxt}(\Gamma)$, a pointed context $\Sigma : \mathrm{Ctxt}(\Gamma, \Theta)$, an enriched context $\Delta : \mathrm{Ty}(\Gamma \vDash \Theta, \Sigma)$ (with enrichment $\overline{\Delta}$, $\sigma : \Delta \to \overline{\Delta}$ and $\delta : 1_\Theta \to \overline{\Delta})$ and a context $\Xi : \mathrm{Ty}(\Gamma \vDash \Theta, \Sigma)$, a type $A : \mathrm{Ty}(\Gamma \vDash \Theta, \Sigma)$, a linear term $u : \mathrm{Tm}(\Gamma \vDash \Theta, \Sigma \mid \Delta \otimes \Xi \vdash A)$ and a linear term $v : \mathrm{Tm}(\Gamma, \Theta \vDash \Sigma \mid \Xi\{\nu_\Theta, \Sigma\} \vdash A\{\nu_\Theta, \Sigma\})$. We say that u maps to v, noted $u \mapsto v$, if there exists a linear term $t : \overline{\Delta} \otimes \Xi \to A$, necessary unique, making the following diagram commute



♣

Given $u : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash A)$, and $v : \mathrm{Tm}(\Gamma \vDash \Theta \vdash A)$, we say that $u \mapsto v$ if there exists a t (necessarily unique) such that



3.6. Semantic type formers

We have not formally defined semantic counterparts to every syntactic constructor, as well as an interpretation for the Hanaba system into an Hanaba model, but a partial attempt can be read at Appendix D.

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Appendix

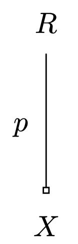
Contents

A Grothendieck fibrations

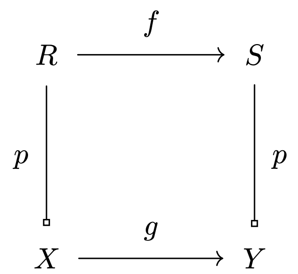
For this section, assume we have a functor $p : \mathcal{E} \to \mathcal{B}$.

A.1 First definitions and notations

For $R : \mathcal{E}$ and $X : \mathcal{B}$, we say that R refines X if $p(R) = X$, and we denote it by



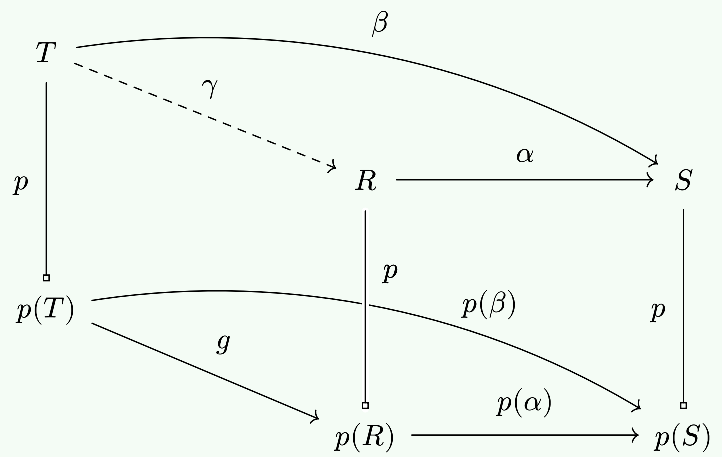
We can omit the p if it is clear from context. Furthermore, we say the following square commutes



if $p(f) = g$.

Definition A.1.1 (Cartesian morphism)

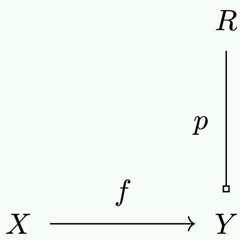
A morphism $\alpha : R \to S$ in $\mathcal{E}$ is said to be *cartesian* if, for any morphism $\beta : T \to S$ in $\mathcal{E}$, and $g : p(T) \to p(R)$ such that $p(\beta) = p(\alpha) \circ g$, there exists a unique $\gamma : T \to R$ making the following diagram commute



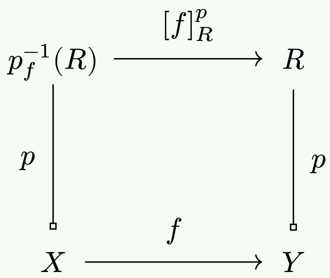
♣

Definition A.1.2 (Grothendieck fibration)

p is a *(Grothendieck) fibration* if for every diagram of the shape



there exists an object $p_f^{-1}(R)$ in $\mathcal{E}$ and a cartesian morphism $[f]_R^p : p_f^{-1}(R) \to R$ making the following diagram commute



Definition A.1.3 (Cloven fibration)

If p is a fibration, p equipped with a choice of such $p_f^{-1}(R)$ and $[f]_R^p$ for every f is called a *cloven fibration.*

Definition A.1.4 (Split fibration)

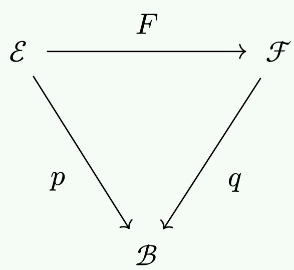
A cloven fibration p is said to be *split* if

$$p_{\mathrm{id}_X}^{-1}(R) = R$$
$$[\mathrm{id}_X]_R^p = \mathrm{id}_R$$

for every R refining X.

Definition A.1.5 (Fibration morphism)

Given two fibrations $p : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{F} \to \mathcal{B}$, a *fibration morphism* $F : p \to q$ is a functor $F : \mathcal{E} \to \mathcal{F}$ making the following diagram commute



and such that any cartesian morphism α in $\mathcal{E}$ is sent to a cartesian morphism $F(\alpha)$ in $\mathcal{F}$.

A.2 Some properties of fibrations

Proposition A.2.1 (Folklore)

The pullback of a fibration by any functor is a fibration.

Proof. Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration, and $q : \mathcal{C} \to \mathcal{B}$ be a functor, and consider the following pullback

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}} \mathcal{E} & \xrightarrow{\pi_2} & \mathcal{E} \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle p} \\ \mathcal{C} & \xrightarrow{q} & \mathcal{B} \end{array}$$

We have to show that π_1 is a fibration. Consider such a diagram

$$\begin{array}{ccc} & & (Y, S) \\ & & \big\downarrow{\scriptstyle \pi_1} \\ X & \xrightarrow{f} & Y \end{array}$$

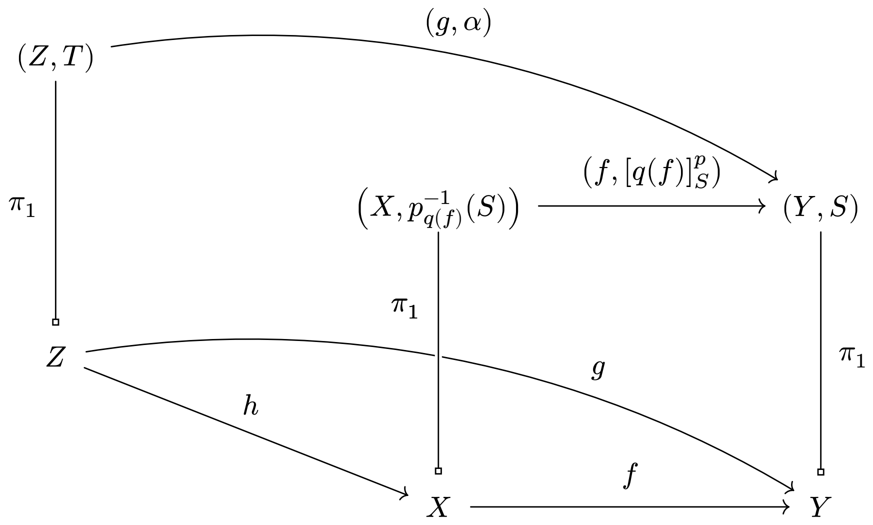
We have that $q(Y) = p(S)$, so we can lift the following diagram

$$\begin{array}{ccc} p^{-1}_{q(f)}(S) & \xrightarrow{[q(f)]^p_S} & S \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ q(X) & \xrightarrow{q(f)} & q(Y) \end{array}$$

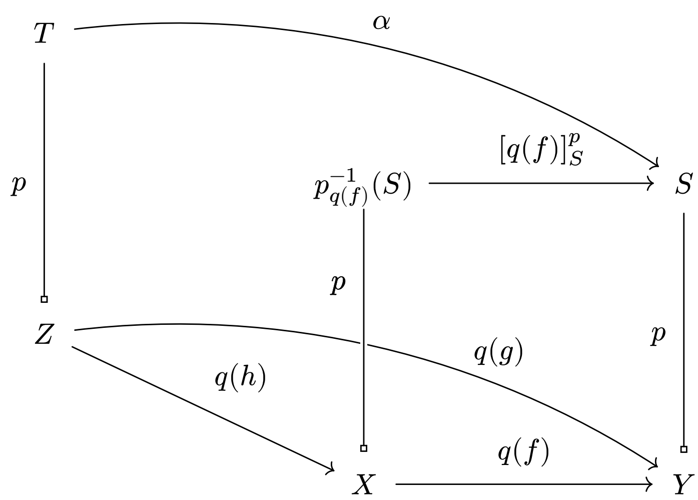
therefore, the following diagram commutes

$$\begin{array}{ccc} \left(X, p^{-1}_{q(f)}(S)\right) & \xrightarrow{(f, [q(f)]^p_S)} & (Y, S) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ X & \xrightarrow{f} & Y \end{array}$$

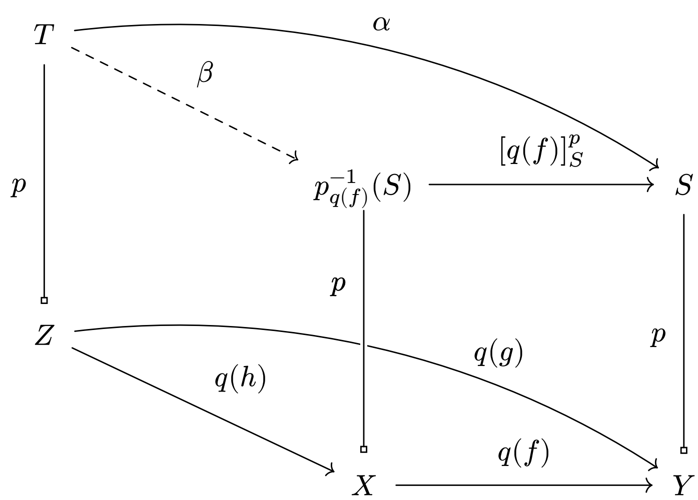
Let us show that this is indeed a cartesian morphism. Let $(g, \alpha) : (Z, T) \to (Y, S)$ be a morphism in $\mathcal{C} \times_{\mathcal{B}} \mathcal{E}$ and $h : Z \to X$ such that the following diagram commutes



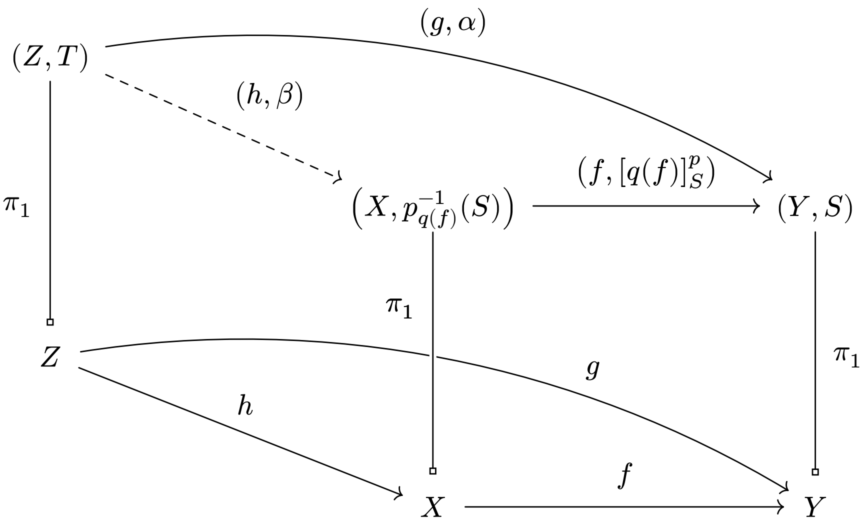
In particular, the following diagram commutes



so there exists a unique $\beta : T \to p^{-1}_{q(f)}(S)$ making the following diagram commute



and so the (h, β) uniquely makes the following diagram commute

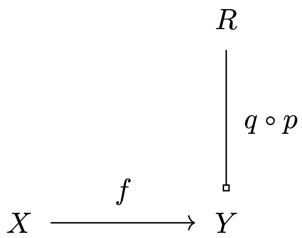


□

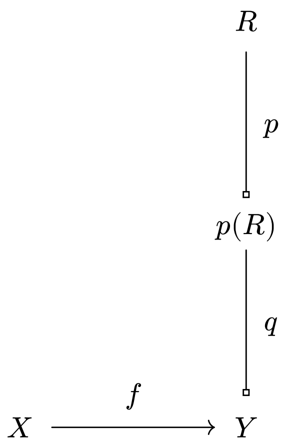
Proposition A.2.2 (Folklore)

Let $p : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{B} \to \mathcal{C}$ two fibrations. $q \circ p$ is a fibration.

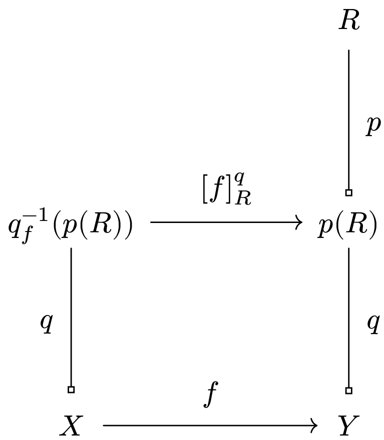
Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$, and R such that $q(p(R)) = Y$.



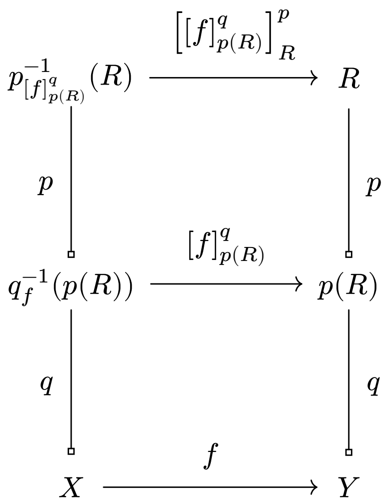
In particular, we are in the following situation



We can lift f one layer:



and again

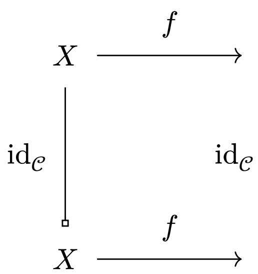


□

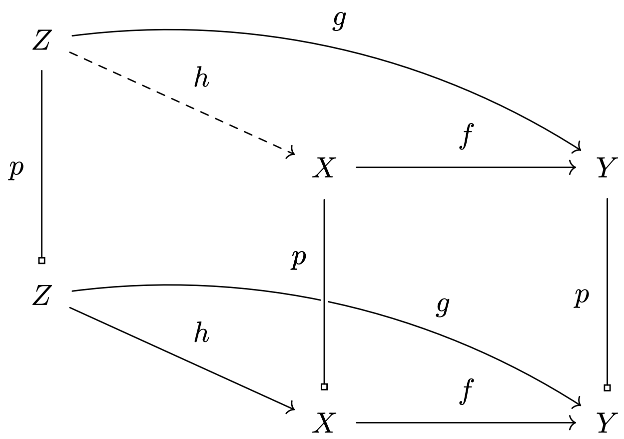
Proposition A.2.3 (Folklore)

Let $\mathcal{C}$ be a category. $\mathrm{id}_{\mathcal{C}}$ is a fibration.

Proof. Consider a morphism $f : X \to Y$. The following diagram trivially commutes



Let us check that f is a cartesian morphism. Let $g : Z \to Y$ be a morphism, and $h : Z \to X$ such that $g = f \circ h$. Then h is the unique morphism making the following diagram commute



□

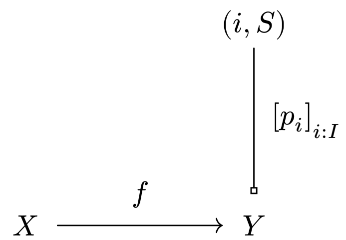
Proposition A.2.4 (Folklore)

Consider a family $(p_i)_{i:I}$ of fibrations $p_i : \mathcal{E}_i \to \mathcal{B}$. Its coproduct

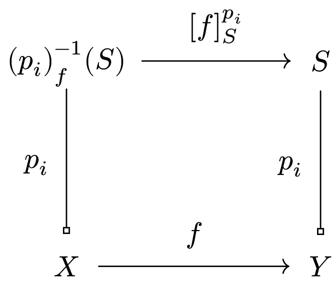
$$[p_i]_{i:I} : \coprod_{i:I} \mathcal{E}_i \longrightarrow \mathcal{B}$$

is a fibration.

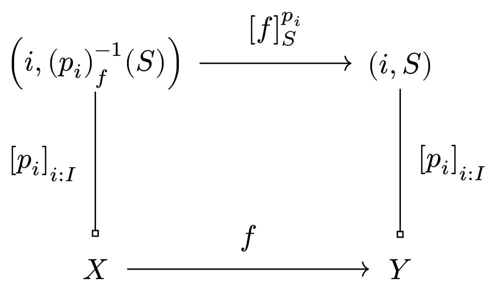
Proof. Consider the following situation



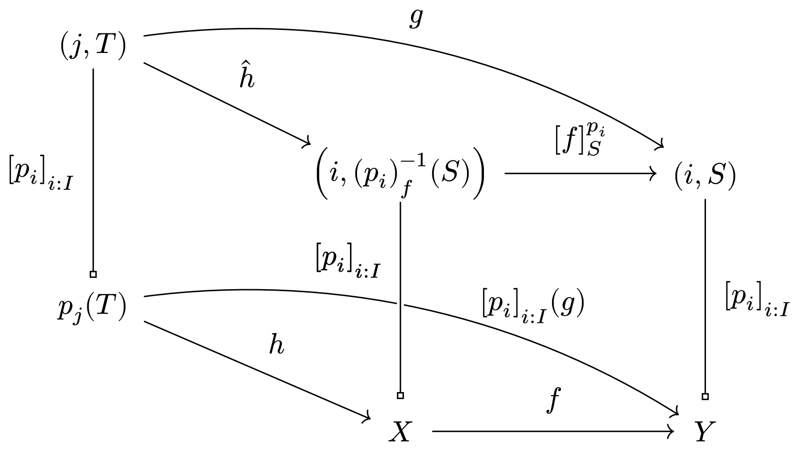
for a given $i : I$ and $S : \mathcal{E}_i$. We can lift f in $\mathcal{E}_i$:



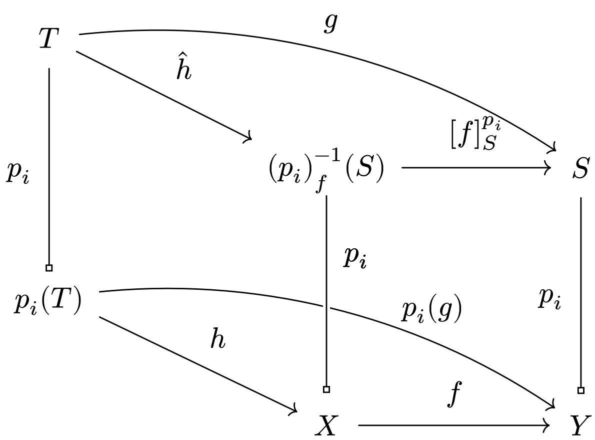
and therefore, in $\coprod_{i:I} \mathcal{E}_i$



Let us show that this morphism is cartesian. Consider a $\hat{h} : (j, T) \to \left(i, (p_i)_f^{-1}(S)\right)$ making the following diagram commute



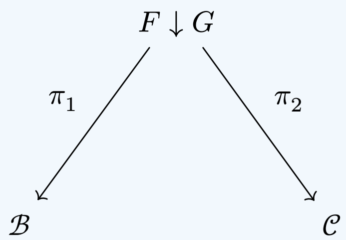
Then we must have $i = j$ and this is exactly equivalent to the commutation of the following diagram



by cartesianity of $[f]_S^{p_i}$, there is exactly one such $\hat{h}$. □

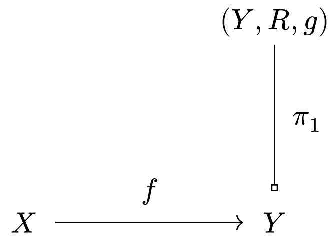
Proposition A.2.5 (Folklore?)

Let $F : \mathcal{B} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. Consider their glueing

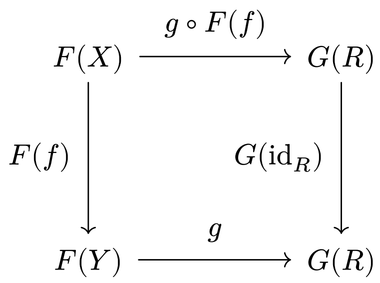


π_1 is a fibration.

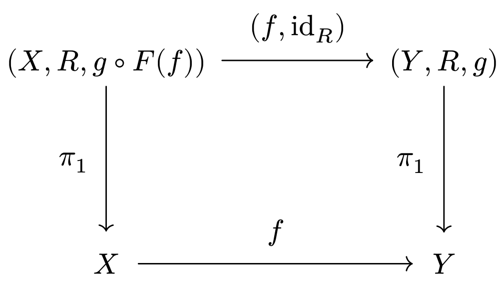
Proof. Let $f : X \to Y$ be a morphism in $\mathcal{B}$, and let $(Y, R, g : F(Y) \to G(R))$ be an object in $F \downarrow G$, such that we are in the following situation



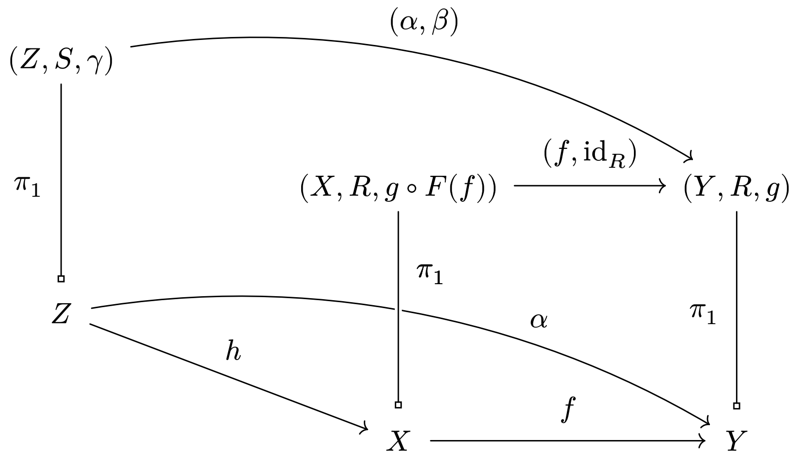
Note that $g \circ F(f) : F(X) \to G(R)$, and that the following diagram commutes



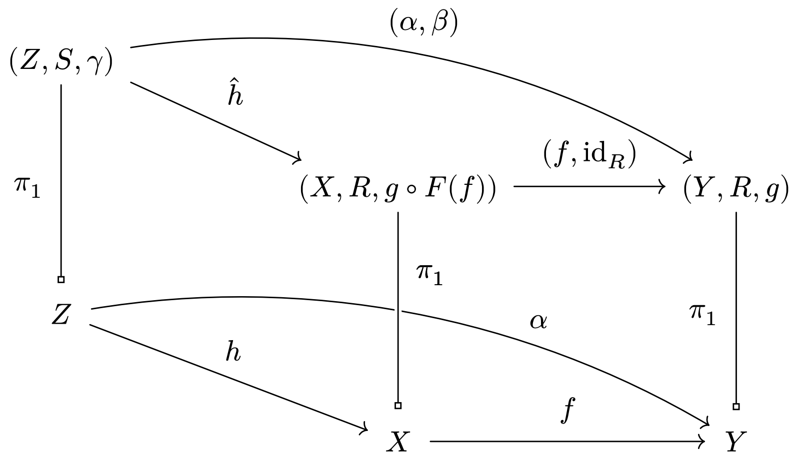
Hence the following diagram commutes



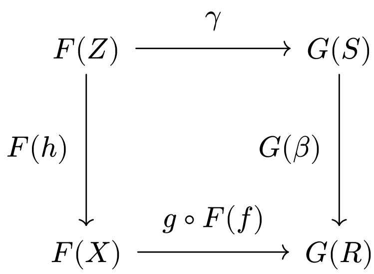
Let us prove that (f, id_R) is cartesian. Suppose we have a $(\alpha, \beta) : (Z, S, \gamma : F(Z) \to G(S))$ as well as an $h : Z \to X$ making the following diagram commute



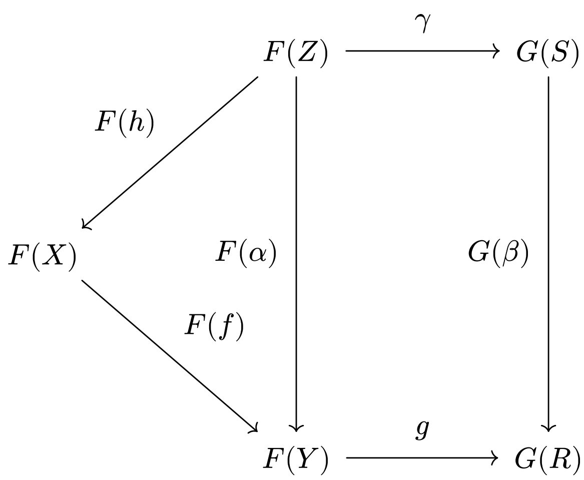
Suppose there is a $\hat{h} : (Z, S, \gamma) \to (X, R, g \circ F(f))$ making the following diagram commute



Then in particular we must have $\hat{h} = (h, \beta)$, so it is unique. Let us check that this indeed forms a morphism $(Z, S, \gamma) \to (X, R, g \circ F(f))$, that is, that the following diagram commutes



It does, as we can decompose it as follows



The left triangle commutes by assumption, and the square commutes because (α, β) is a morphism $(Z, S, \gamma) \to (Y, R, g)$ □

A.3 Grothendieck construction

Theorem A.3.1 (Grothendieck construction [17])

The category of fibrations on a base category $\mathcal{B}$ and fibration morphisms is equivalent to the category of pseudo **Cat**-valued presheaves

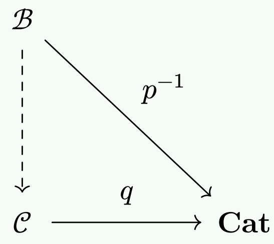
$$\mathcal{B}^{\mathsf{op}} \longrightarrow \mathbf{Cat}$$

For a fibration p over $\mathcal{B}$, we note p^{-1} its corresponding pseudo-functor.

Proof. See Appendix J for a proof. ☐

Definition A.3.2 ($\mathcal{C}$-fibered fibration)

Let $\mathcal{C}$ be a category, and $q : \mathcal{C} \to \mathbf{Cat}$ a pseudo-functor. $p : \mathcal{E} \to \mathcal{B}$ is a $\mathcal{C}$-fibered fibration if it is a fibration, and if p^{-1} factors as such



♣

B Hanaba calculus

B.1 Terms

$$
\begin{aligned}
t ::= {} & x \\
& \mid \mathbb{N} \mid 0 \mid \mathrm{S} \mid R^{\mathbb{N}} \\
& \mid t \multimap t \mid \lambda x.t \mid tt \\
& \mid \forall_{x:t} t \mid \Lambda x.t \mid t@t \\
& \mid t \otimes t \mid (t, t) \mid \text{let } (x, x) = t \text{ in } t \\
& \mid \exists_{x:t} t \mid \langle t, t \rangle \mid \text{let } \langle x, x \rangle = t \text{ in } t \\
& \mid \{t\}_t \mid \text{loop } t \mid \text{let loop } x = t \text{ in } t \mid \text{let } x = \text{lift } x \text{ in } t \\
& \mid t =_t t \mid \text{refl} \mid \mathrm{J} \\
& \mid !t \mid \text{let* } x = t \text{ in } \ldots \text{ let* } x = t \text{ in box } t \mid \text{unbox } t \mid \text{diag } t \mid \text{drop } t; t \\
& \mid t \& t \mid [t, t] \mid \pi_1 \mid \pi_2 \\
& \mid t \oplus t \mid \iota_1 \mid \iota_2 \mid \mathrm{R}^{\oplus} \\
& \mid 1 \mid * \mid t; t \\
& \mid \text{Type}
\end{aligned}
$$

B.2 Rewriting rules

$$\frac{}{(\lambda x.t)u \longrightarrow t[x \mapsto u]}\ \multimap\text{-}\beta \qquad \frac{}{R^{\mathbb{N}}(P, u_0, u_{\text{succ}}, 0) \longrightarrow u_0}\ \mathbb{N}\text{-}\beta\text{-zero}$$

$$\frac{}{(\Lambda x.t)@u \longrightarrow t}\ \forall\text{-}\beta \qquad \frac{}{R^{\mathbb{N}}(P, u_0, u_{\text{succ}}, \mathrm{S}n) \longrightarrow u_{\text{succ}}(n, R^{\mathbb{N}}(P, u_0, u_{\text{succ}}, n))}\ \mathbb{N}\text{-}\beta\text{-succ}$$

$$\frac{}{\text{let } (x, y) = (u, v) \text{ in } t \longrightarrow t[x \mapsto u, y \mapsto v]}\ \otimes\text{-}\beta \qquad \frac{}{\text{let } \langle x, y \rangle = \langle u, v \rangle \text{ in } t \longrightarrow t[y \mapsto v]}\ \exists\text{-}\beta$$

$$\frac{}{\text{let loop } x = \text{loop } u \text{ in } v \longrightarrow v[x \mapsto u]}\ \{\}\text{-}\beta \qquad \frac{}{\text{let } x = \text{lift } u \text{ in } v \longrightarrow v}\ \text{lift-}\beta$$

$$\frac{}{\mathrm{J}(T, P, t, (\text{refl}(T, u, v))) \longrightarrow tu}\ =\text{-}\beta \qquad \frac{}{\text{unbox (let* } x_i = e_i \text{ in box } t) \longrightarrow t[x_i \mapsto e_i]}\ !\text{-}\beta$$

$$\frac{}{\text{diag (let* } x_1 = e_1 \text{ in } \ldots \text{ let* } x_n = e_n \text{ in box } t) \longrightarrow \hat{t}}\ !\text{-diag} \qquad \frac{}{\pi_1[u, v] \longrightarrow u}\ \&\text{-}\beta\text{-}1$$

$$\frac{}{\text{drop (let* } x_i = e_i \text{ in box } t); u \longrightarrow \text{drop } e_1; \ldots; \text{drop } e_n; u}\ !\text{-drop} \qquad \frac{}{\pi_2[u, v] \longrightarrow v}\ \&\text{-}\beta\text{-}2$$

$$\frac{}{\mathrm{R}^{\oplus}(P, \iota_1 @A@Bv, u_1, u_2) \longrightarrow u_1 v}\ \otimes\text{-}\beta\text{-}1 \qquad \frac{}{\mathrm{R}^{\oplus}(P, \iota_2 @A@Bv, u_1, u_2) \longrightarrow u_2 v}\ \otimes\text{-}\beta\text{-}2$$

$$\frac{}{*; u \longrightarrow u}\ 1\text{-}\beta$$

$$\frac{}{\begin{array}{l} \text{let* } x_1 = e_1 \text{ in} \\ \ldots \\ \text{let* } x_n = e_n \text{ in} \\ t \\ \\ \\ \\ \end{array} \longrightarrow \begin{array}{l} \text{let* } x_1 = e_1 \text{ in} \\ \ldots \\ \text{let* } x_{i-1} e = e_{i-1} \text{ in} \\ \text{let* } x_{i+1} = e_{i+1} \text{ in} \\ \ldots \\ \text{let* } x_n = e_n \text{ in} \\ t[x_i \mapsto e_i] \end{array}}$$

where

$$\begin{aligned}\hat{t} = {} & \text{let } (x_1', x_1'') = \text{diag } e_1 \text{ in} \\ & \dots \\ & \text{let } (x_n', x_n'') = \text{diag } e_n \text{ in} \\ & (\text{let* } x_1 = x_1' \text{ in } \dots \text{ let* } x_n = x_n' \text{ in box } t, \text{let* } x_1 = x_1'' \text{ in } \dots \text{ let* } x_n = x_n'' \text{ in box } t)\end{aligned}$$

C Hanaba typing system

We note

$$\Pi_{a:A} B := \forall_{a:A} \{a\}_A \multimap B$$
$$\Sigma_{a:A} B := \exists_{a:A} \{a\}_A \& B$$

C.1 Context formation

$$\frac{\vdash \Gamma \text{ ctxt}}{\Gamma \vdash \diamond \text{ ctxt}}\text{Ctxt-L-empty} \qquad \frac{}{\vdash \diamond \text{ ctxt}}\text{Ctxt-NL-empty}$$

$$\frac{\Gamma \vdash \Delta \text{ ctxt} \quad \Gamma \vdash A \text{ type}}{\Gamma \vdash \Delta, x : A \text{ ctxt}}\text{Ctxt-L-cons} \qquad \frac{\Gamma \vdash \Delta \text{ ctxt} \quad \Gamma \vDash \diamond \mid \diamond \vdash y : A}{\Gamma \vdash \Delta, x \mapsto y : A \text{ ctxt}}\text{Ctxt-L-econs}$$

$$\frac{\vdash \Gamma \text{ ctxt} \quad \Gamma \vdash A \text{ type}}{\vdash \Gamma, x : A \text{ ctxt}}\text{Ctxt-NL-cons}$$

C.2 Universe

$$\frac{}{\Gamma \vdash \text{Type type}}\text{Type-F} \qquad \frac{\Gamma, \Theta \vdash A \text{ type}}{\Gamma \vDash \Theta \mid \diamond \vdash A : \text{Type}}\text{Type-I}$$

$$\frac{\Gamma, \Theta \vdash A \equiv B \text{ type}}{\Gamma \vDash \Theta \mid \diamond \vdash A \equiv B : \text{Type}}\text{Type-}\equiv \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \text{Type} \mapsto \text{Type} : \text{Type}}\text{Type-}\mapsto$$

C.3 Variables

$$\frac{x : A \in \Gamma \quad \vdash \Gamma, \Theta \text{ ctxt}}{\Gamma \vDash \Theta \mid \diamond \vdash x : A}\text{Ax-NL} \qquad \frac{\Gamma, \Theta \vdash x : A \text{ ctxt}}{\Gamma \vDash \Theta \mid x : A \vdash x : A}\text{Ax-L}$$

$$\frac{\Gamma, \Theta \vdash x \mapsto u : A \text{ ctxt}}{\Gamma \vDash \Theta \mid x \mapsto u : A \vdash x : A}\text{Ax-L'} \qquad \frac{x : A \in \Gamma \quad \Gamma, \Theta, \Sigma \vdash A \text{ type}}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash x \mapsto x : A}\text{AX-}\mapsto\text{-NL}$$

$$\frac{\Gamma, \Theta, \Sigma \vdash x : A \text{ ctxt}}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid x : A \vdash x \mapsto x : A}\text{Ax-}\mapsto\text{-L} \qquad \frac{\Gamma, \Theta, \Sigma \vdash x \mapsto u : A \text{ ctxt}}{\Gamma \vDash \Theta \mid \Sigma \mid x \mapsto u : A \mid \diamond \vdash x \mapsto u : A}\text{Ax-}\mapsto\text{-L'}$$

C.4 Structural rules

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash u : A \quad \Gamma, \Theta \mid \diamond \mid \diamond \vdash A \equiv B : \text{Type}}{\Gamma \vDash \Theta \mid \Delta \vdash u : B}\equiv\text{-Cast-L}$$

$$\frac{u \longrightarrow v \quad \Gamma \vDash \Theta \mid \Delta \vdash u : A}{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : A}\equiv\text{-}\beta \qquad \frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : A \\ \Gamma \vDash \Theta \mid \Delta \vdash v \equiv t : A\end{array}}{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv t : A}\equiv\text{-Trans}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash u : A}{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv u : A}\equiv\text{-Refl} \qquad \frac{\Gamma \vDash \Theta \mid \Delta \vdash v \equiv u : A}{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : A}\equiv\text{-Symm}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \equiv B \text{ type} \\ \Gamma \vdash B \equiv C \text{ type}\end{array}}{\Gamma \vdash A \equiv C \text{ type}}\equiv\text{-Trans-type} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}}\equiv\text{-Refl-type}$$

$$\frac{\Gamma \vdash B \equiv A \text{ type}}{\Gamma \vdash A \equiv B \text{ type}}\equiv\text{-Symm-type} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid x \mapsto y : A \mid \diamond \vdash x \mapsto y : A}\mapsto\text{-Ax}$$

C.5 Natural numbers

$$\frac{}{\Gamma \vdash \mathbb{N} \text{ type}}\,\mathbb{N}\text{-F} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \mathbb{N} \mapsto \mathbb{N} : \text{Type}}\,\mathbb{N}\text{-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash 0 : \mathbb{N}}\,\mathbb{N}\text{-I-zero} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash 0 \mapsto 0 : \mathbb{N}}\,\mathbb{N}\text{-I-zero-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \text{succ} : \mathbb{N} \multimap \mathbb{N}}\,\mathbb{N}\text{-I-succ} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \text{succ} \mapsto \text{succ} : \mathbb{N} \multimap \mathbb{N}}\,\mathbb{N}\text{-I-succ-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash R^{\mathbb{N}} : \forall_{P:\forall_{n:\mathbb{N}} \text{ type}} \Pi_{z:P(0)} \forall_{s:\forall_{n:\mathbb{N}} P(n) \multimap P(\text{succ } n)} !\{s\} \multimap \Pi_{n:\mathbb{N}} P(n)}\,\mathbb{N}\text{-E}$$

$$\frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash R^{\mathbb{N}} \mapsto R^{\mathbb{N}} : \forall_{P:\forall_{n:\mathbb{N}} \text{ type}} \Pi_{z:P(0)} \forall_{s:\forall_{n:\mathbb{N}} P(n) \multimap P(\text{succ } n)} !\{s\} \multimap \Pi_{n:\mathbb{N}} P(n)}\,\mathbb{N}\text{-E-}\mapsto$$

C.6 Singleton type

$$\frac{\Gamma \vDash \diamond \mid \diamond \vdash u : A}{\Gamma \vdash \{u\}_A \text{ type}}\,\{\}\text{-F} \qquad \frac{\Gamma \vDash \Theta \mid \Delta \vdash a \equiv b : A \qquad \Gamma, \Theta \mid \diamond \mid \diamond \vdash A \equiv B : \text{Type}}{\Gamma \vDash \Theta \mid \Delta \vdash \{a\}_A \equiv \{b\}_B : \text{Type}}\,\{\}\text{-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \diamond \mid \Delta \mid \diamond \vdash t \mapsto a : A}{\Gamma \vDash \Theta \mid \Delta \vdash \text{loop } t : \{a\}_A}\,\{\}\text{-I} \qquad \frac{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : A \qquad \Gamma \vDash \Theta \mid \diamond \mid \Delta \mid \diamond \vdash u \mapsto a : A}{\Gamma \vDash \Theta \mid \Delta \vdash \text{loop } u \equiv \text{loop } v : \{a\}_A}\,\{\}\text{-I-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \diamond \vdash u \mapsto v : A \qquad \Gamma \vDash \Theta, \Sigma \mid \Delta \vdash u \mapsto a : A}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \diamond \vdash \text{loop } u \mapsto \text{loop } v : \{a\}_A}\,\{\}\text{-I-}\mapsto$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_2, x \mapsto a : A \vdash v : B \qquad \Gamma \vDash \Theta \mid \Delta_1 \vdash u : \{a\}_A}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let loop } x = u \text{ in } v : B}\,\{\}\text{-E-L}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u \equiv u' : \{a\}_A \qquad \Gamma \vDash \Theta \mid \Delta_2, x \mapsto a : A \vdash v \equiv v' : B}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let loop } x = u \text{ in } v \equiv \text{let loop } x = u' \text{ in } v' : B}\,\{\}\text{-E-L-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1 \mid \Xi_1 \vdash u : \{a\}_A \qquad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_2, x \mapsto a : A \mid \Xi_2 \vdash v \mapsto v'}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Delta_2 \mid \Xi_1, \Xi_2 \vdash \text{let loop } x = u \text{ in } v \mapsto v'}$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Xi_1 \vdash u \mapsto u' : \{a\}_A \qquad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_2, x \mapsto a : A \mid \Xi_2 \vdash v \mapsto v'[x \mapsto a] : B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Delta_2 \mid \Xi_1, \Xi_2 \vdash \text{let loop } x = u \text{ in } v \mapsto \text{let loop } x = u' \text{ in } v' : B}\,\{\}\text{-E-}\mapsto$$

$$\frac{\Gamma \vDash \Theta, x : C \mid \Delta_1, y \mapsto x : C, \Delta_2 \vdash t : A}{\Gamma \vDash \Theta \mid \Delta_1, y : C, \Delta_2 \vdash \text{let } x = \text{lift } y \text{ in } t : A}\,\text{Lift} \qquad \frac{}{\Gamma \vDash \Delta \vdash}\,\text{Lift-}\equiv$$

$$\frac{\Gamma \vDash \Theta, y : B \mid \Sigma \mid \Delta, x \mapsto y : B \mid \Xi_1, \Xi_2 \vdash u \mapsto v[x \mapsto y] : A}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi_1, x : B, \Xi_2 \vdash \text{let } y = \text{lift } x \text{ in } u \mapsto \text{let } y = \text{lift } x \text{ in } v : A}\,\text{Lift-}\mapsto\text{-1}$$

$$\frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, x : B, \Delta_2 \mid \Xi \vdash \text{let } y = \text{lift } x \text{ in } u \mapsto \text{let } y = \text{lift?} \text{ in } v : A}\,\text{Lift-}\mapsto\text{-2}$$

C.7 Identity type

$$\frac{\Gamma \vDash \diamond \mid \diamond \vdash u : A \qquad \Gamma \vDash \diamond \mid \diamond \vdash v : A}{\Gamma \vdash u =_A v \text{ type}}\,\text{Id-F} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type} \qquad \Gamma \vDash \diamond \mid \diamond \vdash u \equiv u' : A \qquad \Gamma \vDash \diamond \mid \diamond \vdash v \equiv v' : A}{\Gamma \vdash u =_A v \equiv u' =_{A'} v' \text{ type}}\,\text{Id-}\equiv$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \mathrm{refl} : \Pi_{a:A} a =_A a}\text{Id-I} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \mathrm{refl} \mapsto \mathrm{refl} : \Pi_{a:A} a =_A a}\text{Id-I-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash J : \forall_{A:\ \mathrm{Type}} \forall_{P: \forall_{a:A} \forall_{b:A} \forall_{e:a=_A b}\ \mathrm{type}} (\Pi_{x:A} P(x, x, \mathrm{refl}(A, x, x))) \multimap \forall_{a:A} \forall_{b:A} \Pi_{e:a=_A b} P(a, b, e)}\text{Id-E}$$

$$\frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash J \mapsto J : \forall_{A:\ \mathrm{Type}} \forall_{P: \forall_{a:A} \forall_{b:A} \forall_{e:a=_A b}\ \mathrm{type}} (\Pi_{x:A} P(x, x, \mathrm{refl}(A, x, x))) \multimap \forall_{a:A} \forall_{b:A} \Pi_{e:a=_A b} P(a, b, e)}\text{Id-E-}\mapsto$$

C.8 Lollipop

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \multimap B \text{ type}}\multimap\text{-F} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash B \equiv B' \text{ type}}{\Gamma \vdash A \multimap B \equiv A' \multimap B' \text{ type}}\multimap\text{-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Delta, x : A \vdash t : B}{\Gamma \vDash \Theta \mid \Delta \vdash \lambda x.t : A \multimap B}\multimap\text{-I} \qquad \frac{\Gamma \vDash \Delta, x : A \vdash u \equiv v : B}{\Gamma \vDash \Delta \vdash (\lambda x.u) \equiv (\lambda x.v) : A \multimap B}\multimap\text{-I-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi, x : A \vdash u \mapsto v : B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \lambda x.u \mapsto \lambda y.v : A \multimap B}\multimap\text{-I-}\mapsto \qquad \frac{\Gamma \vDash \Delta_1 \vdash t : A \multimap B \quad \Gamma \vDash \Delta_2 \vdash u : A}{\Gamma \vDash \Delta_1, \Delta_2 \vdash tu : B}\multimap\text{-E}$$

$$\frac{\begin{array}{c}\Gamma \vDash \Delta_1 \vdash u \equiv u' : A \multimap B \\ \Gamma \vDash \Delta_2 \vdash v \equiv v' : A\end{array}}{\Gamma \vDash \Delta_1, \Delta_2 \vdash uv \equiv u'v' : B}\multimap\text{-E-}\equiv \qquad \frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1 \mid \Xi_1 \vdash u \mapsto u' : A \multimap B \\ \Gamma \vDash \Theta \mid \Sigma \mid \Delta_2 \mid \Xi_2 \vdash v \mapsto v' : A\end{array}}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Delta_2 \mid \Xi_1, \Xi_2 \vdash uv \mapsto u'v' : B}\multimap\text{-E-}\mapsto$$

C.9 Universal quantifier

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, a : A \vdash B \text{ type}}{\Gamma \vdash \forall_{a:A} B \text{ type}}\forall\text{-F} \qquad \frac{\begin{array}{c}\Gamma \vdash A \equiv A' \text{ type} \\ \Gamma, a : A \vdash B \equiv B' \text{ type}\end{array}}{\Gamma \vdash \forall_{a:A} B \equiv \forall_{a:A'} B' \text{ type}}\forall\text{-}\equiv$$

$$\frac{\Gamma \vDash \Theta, x : A \mid \Delta \vdash t : B}{\Gamma \vDash \Theta \mid \Delta \vdash \Lambda x.t : \forall_{x:A} B}\forall\text{-I} \qquad \frac{\Gamma \vDash \Theta, x : A \mid \Delta \vdash u \equiv v : B}{\Gamma \vDash \Theta \mid \Delta \vdash \Lambda x.u \equiv \Lambda x.v : \forall_{x:A} B}\forall\text{-I-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma, x : A \mid \Delta \mid \Xi \vdash u \mapsto v : B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \Lambda x.u \mapsto \Lambda x.v : \forall_{x:A} B}\forall\text{-I-}\mapsto \qquad \frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Delta \vdash u : \forall_{a:A} B \\ \Gamma, \Theta \mid \diamond \mid \diamond \vdash v : A\end{array}}{\Gamma \vDash \Theta \mid \Delta \vdash u@v : B[a \mapsto v]}\forall\text{-E}$$

$$\frac{\Gamma \vDash \Delta \vdash u \equiv u' : \forall_{a:A} B \quad \Gamma \vdash v \equiv v' : A}{\Gamma \vDash \Delta \vdash u@v \equiv u'@v' : B[a \mapsto v]}\forall\text{-E-L-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto u' : \forall_{a:A} B \quad \Gamma, \Theta, \Sigma \mid \diamond \mid \diamond \mid \diamond \mid \diamond \vdash v \mapsto v' : A}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u@v \mapsto u'@v' : B[a \mapsto v]}\forall\text{-E-}\mapsto$$

C.10 Monoidal product

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \otimes B \text{ type}}\otimes\text{-F} \qquad \frac{\Gamma \vdash A \equiv A' : \mathrm{Type} \quad \Gamma \vdash B \equiv B' : \mathrm{Type}}{\Gamma \vdash A \otimes B \equiv A' \otimes B' : \mathrm{Type}}\otimes\text{-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u : A \quad \Gamma \vDash \Theta \mid \Delta_2 \vdash v : B}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash (u, v) : A \otimes B}\otimes\text{-I} \qquad \frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Delta_1 \vdash u \equiv u' : A \\ \Gamma \vDash \Theta \mid \Delta_2 \vdash v \equiv v' : B\end{array}}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash (u, v) \equiv (u', v') : A \otimes B}\otimes\text{-I-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1 \mid \Xi_1 \vdash u \mapsto u' : A \quad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_2 \mid \Xi_2 \vdash v \mapsto v' : B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Delta_2 \mid \Xi_1, \Xi_2 \vdash (u, v) \mapsto (u', v') : A \otimes B}\otimes\text{-I-}\mapsto$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1, x : A, y : B \vdash t : C \quad \Gamma \vDash \Theta \mid \Delta_2 \vdash u : A \otimes B}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let } (x, y) = u \text{ in } t : C} \otimes\text{-E}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1, x : A, y : B \vdash v \equiv v' : C \quad \Gamma \vDash \Theta \mid \Delta_2 \vdash u \equiv u' : A \otimes B}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let } (x, y) = u \text{ in } v \equiv \text{let } (x, y) = u' \text{ in } v' : C} \otimes\text{-E-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1 \mid \Xi_1, x : A, y : B \vdash v \mapsto v' : C \quad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_2 \mid \Xi_2 \vdash u \mapsto u' : A \otimes B}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Delta_2 \mid \Xi_1, \Xi_2 \vdash \text{let } (x, y) = u \text{ in } v \mapsto \text{let } (x, y) = u' \text{ in } v' : C} \otimes\text{-E-}\mapsto$$

C.11 Existential quantifier

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, a : A \vdash B \text{ type}}{\Gamma \vdash \exists_{a:A} B \text{ type}} \exists\text{-F} \qquad \frac{\begin{array}{c}\Gamma \vdash A \equiv A' \text{ type} \\ \Gamma, a : A \vdash B \equiv B' \text{ type}\end{array}}{\Gamma \vdash \exists_{a:A} B \equiv \exists_{a:A'} B' \text{ type}} \exists\text{-}\equiv$$

$$\frac{\begin{array}{c}\Gamma, \Theta \mid \diamond \mid \diamond \vdash u : A \\ \Gamma \vDash \Theta \mid \Delta \vdash v : B[a \mapsto u]\end{array}}{\Gamma \vDash \Theta \mid \Delta \vdash \langle u, v \rangle : \exists_{a:A} B} \exists\text{-I} \qquad \frac{\begin{array}{c}\Gamma, \Theta \mid \diamond \mid \diamond \vdash u \equiv u' : A \\ \Gamma \vDash \Theta \mid \Delta \vdash v \equiv v' : B[a \mapsto u]\end{array}}{\Gamma \vDash \Theta \mid \Delta \vdash \langle u, v \rangle \equiv \langle u', v' \rangle : \exists_{a:A} B} \exists\text{-I-}\equiv$$

$$\frac{\Gamma, \Theta, \Sigma \mid \diamond \mid \diamond \mid \diamond \mid \diamond \vdash u \mapsto u' : A \quad \Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash v \mapsto v' : B[a \mapsto u]}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \langle u, v \rangle \mapsto \langle u', v' \rangle : \exists_{a:A} B} \exists\text{-I-}\mapsto$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u : \exists_{a:A} B \quad \Gamma \vDash \Theta, x : A \mid \Delta_2, y : B[a \mapsto x] \vdash v : C \quad \Gamma \vDash \Theta \vdash C \text{ type}}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let } \langle x, y \rangle = u \text{ in } v : C} \exists\text{-E}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u \equiv u' : \exists_{a:A} B \quad \Gamma \vDash \Theta, x : A \mid \Delta_2, y : B[a \mapsto x] \vdash v \equiv v' : C}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash \text{let } \langle x, y \rangle = u \text{ in } v \equiv \text{let } \langle x, y \rangle = u' \text{ in } v' : C} \exists\text{-E-}\equiv$$

$$\frac{\Gamma \vDash \Delta_1 \vdash u \mapsto u' : \exists_{a:A} B \quad \Gamma, x : A \mid \Delta_2, y : B[a \mapsto x] \vdash v \mapsto v' : C}{\Gamma \vDash \Delta_1, \Delta_2 \vdash \text{let } \langle x, y \rangle = u \text{ in } v \mapsto \text{let } \langle x, y \rangle = u' \text{ in } v' : C} \exists\text{-E-}\mapsto$$

C.12 Bang modality

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash !A \text{ type}} !\text{-F} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type}}{\Gamma \vdash !A \equiv !A' \text{ type}} !\text{-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid x_1 : !A_1, \ldots, x_n : !A_n \vdash t : A \quad \Gamma \vDash \Theta \mid \Delta_1 \vdash u_1 : !A_1 \quad \ldots \quad \Gamma \vDash \Theta \mid \Delta_n \vdash u_n : !A_n}{\Gamma \vDash \Theta \mid \Delta_1, \ldots, \Delta_n \vdash \text{let* } x_1 = u_1 \text{ in } \ldots \text{ let* } x_n = u_n \text{ in box } t : !A} !\text{-I}$$

$$\frac{\begin{array}{c}\Gamma \vDash \Theta \mid x_1 : !A_1, \ldots, x_n : !A_n \vdash u \equiv v : A \quad \Gamma \vDash \Theta \mid \Delta_1 \vdash u_1 \equiv v_1 : !A_1 \quad \ldots \\ \Gamma \vDash \Theta \mid \Delta_n \vdash u_n \equiv v_n : !A_n\end{array}}{\Gamma \vDash \Theta \mid \Delta_1, \ldots, \Delta_n \vdash \begin{array}{l}\text{let* } x_1 = u_1 \text{ in} \\ \ldots \\ \text{let* } x_n = u_n \text{ in} \\ \text{box } u\end{array} \equiv \begin{array}{l}\text{let* } x_1 = v_1 \text{ in} \\ \ldots \\ \text{let* } x_n = v_n \text{ in} \\ \text{box } v\end{array} : !A} !\text{-I-}\equiv$$

$$\frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid x_1 : !A_1, \ldots, x_n : !A_n \vdash u \mapsto v : A \quad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_1 \mid \Xi_1 \vdash u_1 \mapsto v_1 : !A_1 \\ \ldots \quad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_n \mid \Xi_n \vdash u_n \mapsto v_n : !A_n\end{array}}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \ldots, \Delta_n \mid \Xi_1, \ldots, \Xi_n \vdash \begin{array}{l}\text{let* } x_1 = u_1 \text{ in} \\ \ldots \\ \text{let* } x_n = u_n \text{ in} \\ \text{box } u\end{array} \mapsto \begin{array}{l}\text{let* } x_1 = v_1 \text{ in} \\ \ldots \\ \text{let* } x_n = v_n \text{ in} \\ \text{box } v\end{array} : !A} !\text{-I-}\mapsto$$

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash t : {!A}}{\Gamma \vDash \Theta \mid \Delta \vdash \text{unbox } t : A}\;\text{!-E} \qquad \frac{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : {!A}}{\Gamma \vDash \Theta \mid \Delta \vdash \text{unbox } u \equiv \text{unbox } v : A}\;\text{!-E-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto v : {!A}}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \text{unbox } u \mapsto \text{unbox } v : A}\;\text{!-E-}\mapsto \qquad \frac{\Gamma \vDash \Theta \mid \Delta \vdash t : {!A}}{\Gamma \vDash \Theta \mid \Delta \vdash \text{diag } t : {!A} \otimes {!A}}\;\text{!-diag}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv v : {!A}}{\Gamma \vDash \Theta \mid \Delta \vdash \text{diag } u \equiv \text{diag } v : {!A} \otimes {!A}}\;\text{!-diag-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto v : {!A}}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash \text{diag } u \mapsto \text{diag } v : {!A} \otimes {!A}}\;\text{!-diag-}\mapsto$$

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash u : {!A}}{\Gamma \vDash \Theta \mid \Delta \vdash u \equiv \text{let* } x = u \text{ in box } (\text{unbox } u) : {!A}}\;\text{!-}\eta$$

C.13 With

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \& B \text{ type}}\;\&\text{-F} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash B \equiv B' \text{ type}}{\Gamma \vdash A \& B \equiv A' \& B' \text{ type}}\;\&\text{-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Delta \vdash u : A \quad \Gamma \vDash \Theta \mid \Delta \vdash v : B}{\Gamma \vDash \Theta \mid \Delta \vdash [u, v] : A \& B}\;\&\text{-I} \qquad \frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Delta \vdash u \equiv u' : A \\ \Gamma \vDash \Theta \mid \Delta \vdash v \equiv v' : A\end{array}}{\Gamma \vDash \Theta \mid \Delta \vdash [u, v] \equiv [u', v'] : A \& B}\;\&\text{-I-}\equiv$$

$$\frac{\begin{array}{c}\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto u' : A \\ \Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash v \mapsto v' : A\end{array}}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash [u, v] \mapsto [u', v'] : A \& B}\;\&\text{-I-}\mapsto \qquad \frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \pi_1 : A \& B \multimap A}\;\&\text{-E-1}$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \pi_2 : A \& B \multimap B}\;\&\text{-E-2} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \pi_1 \mapsto \pi_1 : A \& B \multimap A}\;\&\text{-E-1-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \pi_2 \mapsto \pi_2 : A \& B \multimap B}\;\&\text{-E-2-}\mapsto$$

C.14 Coproduct

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \oplus B \text{ type}}\;\oplus\text{-F} \qquad \frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma \vdash B \equiv B' \text{ type}}{\Gamma \vdash A \oplus B \equiv A' \oplus B' \text{ type}}\;\oplus\text{-}\equiv$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \iota_1 : \forall_{A:\,\text{Type}} \forall_{B:\,\text{Type}} A \multimap A \oplus B}\;\oplus\text{-I-1} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \iota_1 \mapsto \iota_1 : \forall_{A:\,\text{Type}} \forall_{B:\,\text{Type}} A \multimap A \oplus B}\;\oplus\text{-I-1-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \iota_2 : \forall_{A:\,\text{Type}} \forall_{B:\,\text{Type}} B \multimap A \oplus B}\;\oplus\text{-I2} \qquad \frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \iota_2 \mapsto \iota_2 : \forall_{A:\,\text{Type}} \forall_{B:\,\text{Type}} B \multimap A \oplus B}\;\oplus\text{-I2-}\mapsto$$

$$\frac{}{\Gamma \vDash \Theta \mid \diamond \vdash \mathrm{R}^{\oplus} : \forall_{A:\,\text{Type}} \forall_{B:\,\text{Type}} \forall_{P : \forall_{x:A \oplus B} \text{Type}} (\Pi_{a:A} P(\iota_1 a)) \multimap (\Pi_{b:B} P(\iota_2 b)) \multimap \Pi_{x:A \oplus B} P(x)}\;\oplus\text{-E}$$

$$\frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash \mathrm{R}^{\oplus} \mapsto \mathrm{R}^{\oplus} : \forall_{A:\,\text{Type}} \forall_{B:\,\text{Type}} \forall_{P : \forall_{x:A \oplus B} \text{Type}} (\Pi_{a:A} P(\iota_1 a)) \multimap (\Pi_{b:B} P(\iota_2 b)) \multimap \Pi_{x:A \oplus B} P(x)}\;\oplus\text{-E-}\mapsto$$

C.15 Unit

$$\frac{}{\Gamma \vdash 1 \text{ type}}\;\text{1-F} \qquad \frac{}{\Gamma \vDash \Theta \mid \diamond \vdash * : 1}\;\text{1-I}$$

$$\frac{}{\Gamma \vDash \Theta \mid \Sigma \mid \diamond \mid \diamond \vdash * \mapsto * : 1}\;\text{1-I-}\mapsto \qquad \frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u : 1 \quad \Gamma \vDash \Theta \mid \Delta_2 \vdash v : A}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash u; v : A}\;\text{1-E}$$

$$\frac{\Gamma \vDash \Theta \mid \Delta_1 \vdash u \equiv u' : 1 \qquad \Gamma \vDash \Theta \mid \Delta_2 \vdash v \equiv v' : A}{\Gamma \vDash \Theta \mid \Delta_1, \Delta_2 \vdash u; v \equiv u'; v' : A} \text{1-E-}\equiv$$

$$\frac{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1 \mid \Xi_1 \vdash u \mapsto u' : 1 \qquad \Gamma \vDash \Theta \mid \Sigma \mid \Delta_2 \mid \Xi_2 \vdash v \mapsto v' : A}{\Gamma \vDash \Theta \mid \Sigma \mid \Delta_1, \Delta_2 \mid \Xi_1, \Xi_2 \vdash u; v \mapsto u'; v' : A} \text{1-E-}\mapsto$$

%

D Semantic type formers

What we have defined so far forms the backbone of Hanaba models. In order to interpret the syntax, we additionally need to have semantic counterparts to type formers.

D.1 Linear implication

A Habana model supports linear implication if the category $\mathsf{Ty}_l(\Gamma \vDash \Theta)$ is monoidal *closed*, that is, there exists, for every $A, B : \mathsf{Ty}_l(\Gamma \vDash \Theta)$, an object $A \multimap B$ (16) , and natural bijections

$$\mathsf{Ty}_l(\Gamma \vDash \Theta)(A \otimes B, C) \underset{\mathrm{ev}}{\overset{\lambda}{\rightleftarrows}} \mathsf{Ty}_l(\Gamma \vDash \Theta)(A, B \multimap C)$$

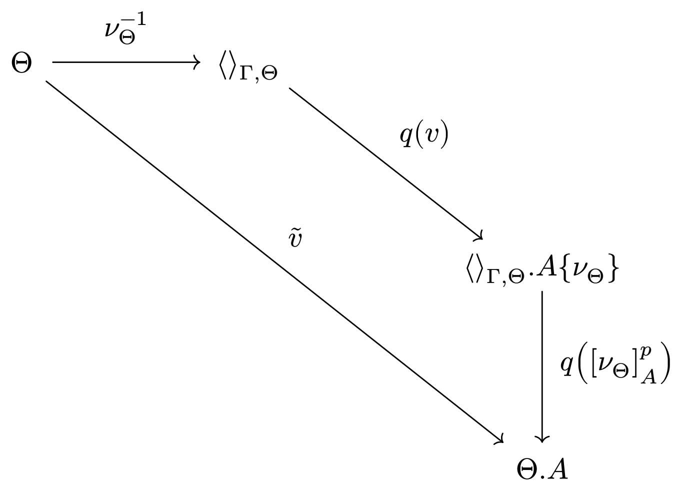
such that $\{\!\!\{-\}\!\!\}_{\Gamma\vDash\Theta}$, restricted to $\mathsf{Ty}_l(\Gamma \vDash \Theta)$, is a monoidal closed functor.

D.2 Universal quantification

A Habana model supports universal quantification if, for any global context $\Gamma : \mathcal{B}$, for $\Theta : \mathrm{Ctxt}(\Gamma)$, for $A : \mathrm{Ty}(\Gamma \vDash \Theta)$ and $B : \mathrm{Ty}(\Gamma \vDash \Theta.A)$, there exists a type $\forall(A, B) : \mathrm{Ty}(\Gamma \vDash \Theta)$ such that, for any term $u : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash \forall(A, B))$, and a term $v : \mathrm{Tm}\big(\Gamma, \Theta \vDash \langle\rangle_{\Gamma,\Theta} \mid 1_{\langle\rangle_{\Gamma,\Theta}} \vdash A\{\nu_\Theta\}\big)$, there is a term $\mathrm{At}(u, v) : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash B\{\tilde{v}\})$ (17) and, for any term $u : \mathrm{Tm}(\Gamma \vDash \Theta.A \mid \Delta\{\overline{\mathbf{p}}_A\} \vdash B)$, there is a term $\Lambda(u) : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash \forall(A, B))$ (18) such that

$$\mathrm{At}(\Lambda(u), v) = u\{\tilde{v}\}$$

where



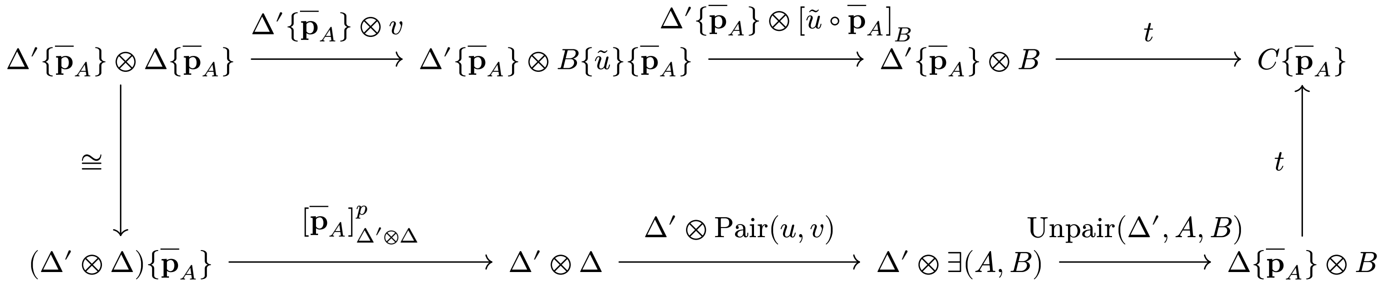
Additionally, these constructors must be compatible with substitutions and with the $\mapsto$ relation.

D.3 Existential quantification

A Hanaba model supports existential quantification if, for any global context $\Gamma : \mathcal{B}$, for $\Theta : \mathrm{Ctxt}(\Gamma)$, for $A : \mathrm{Ty}(\Gamma \vDash \Theta)$ and $B : \mathrm{Ty}(\Gamma \vDash \Theta.A)$, there exists a type $\exists(A, B) : \mathrm{Ty}(\Gamma \vDash \Theta)$ such that, for any term $u : \mathrm{Tm}\big(\Gamma, \Theta \vDash \langle\rangle_{\Gamma,\Theta} \mid 1_{\langle\rangle_{\Gamma,\Theta}} \vdash A\{\nu_\Theta\}\big)$ and a term $v : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash B\{\tilde{u}\})$, there is a term $\mathrm{Pair}(u, v) : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash \exists(A, B))$ (19) and, for any $\Delta : \mathrm{Ty}(\Gamma \vDash \Theta)$, there is a morphism $\mathrm{Unpair}(\Delta, A, B)$ (20) as follows:

$$\mathrm{Unpair}(\Delta, A, B) : \Delta \otimes \exists(A, B) \longrightarrow (\Delta\{\overline{\mathbf{p}}_A\}) \otimes B$$

such that, for any type $C : \mathrm{Ty}(\Gamma \vDash \Theta)$ and a term $t : \mathrm{Tm}(\Gamma \vDash \Theta.A \mid \Delta'\{\overline{\mathbf{p}}_A\} \otimes B \vdash C\{\overline{\mathbf{p}}_A\})$, we have that the following diagram commutes

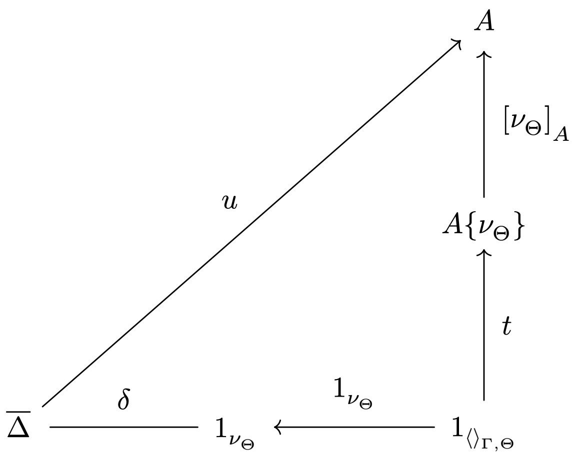


The commutation of this diagram ensures that the β rule for existential types is respected.

D.4 Singleton type

Let $\Gamma : \mathcal{B}$, $\Theta : \mathrm{Ctxt}(\Gamma)$, $A : \mathrm{Ty}(\Gamma \vDash \Theta)$. For any linear term $t : \mathrm{Tm}\big(\Gamma, \Theta \vDash \langle\rangle_{\Gamma,\Theta} \mid 1_{\langle\rangle_{\Gamma,\Theta}} \vdash A\{\nu_\Theta\}\big)$, assume there is a type $\mathrm{Sgl}(t) : \mathrm{Ty}(\Gamma \vDash \Theta)$ (21) .

Furthermore, assume that, for any any enriched context $\big(\Delta, \overline{\Delta}, \sigma, \delta\big)$, and for any $u : \mathrm{Tm}\big(\Gamma \vDash \Theta \mid \overline{\Delta} \vdash A\big)$ making the following diagram commute



there is a term $\mathrm{Loop}(u)$ (22) :

$$\mathrm{Loop}(u, t) : \mathrm{Tm}(\Gamma \vDash \Theta \mid \Delta \vdash \mathrm{Sgl}(t))$$

D.5 Interpreting the syntax

Given a Hanaba model, we define in a mutual recursive way the following:

- given a (syntactic) context Γ such that $\vdash \Gamma$ ctxt, we interpret it as a global context $[\![\Gamma]\!] : \mathcal{B}$
- given a (syntactic) context Θ such that $\vdash \Gamma, \Theta$ ctxt, we interpret it as $[\![\Gamma \vDash \Theta]\!] : \mathrm{Ctxt}([\![\Gamma]\!])$
- given a (syntactic) annotated context Δ such that $\Gamma, \Theta \vdash \Delta$ ctxt, we interpret it as an enriched linear context $[\![\Delta]\!] : \mathrm{Ty}([\![\Gamma]\!] \vDash [\![\Theta]\!])$, where $\overline{[\![\Delta]\!]} = [\![\overline{\Delta}]\!]$, with $\overline{\Delta}$ being define as follows:

$$\overline{\diamond} = \diamond$$
$$\overline{(\Delta, x : A)} = \overline{\Delta}$$
$$\overline{(\Delta, x \mapsto a : A)} = \overline{\Delta}, x : A$$

- given a (syntactic) type A such that $\Gamma, \Theta \vdash A$ type, we interpret it as $[\![A]\!] : \mathrm{Ty}([\![\Gamma]\!] \vDash [\![\Theta]\!])$
- given a (syntactic) context Ξ such that $\Gamma, \Theta \vdash \Xi$ ctxt, we interpret it as $[\![\Xi]\!] : \mathrm{Ty}([\![\Gamma]\!] \vDash [\![\Theta]\!])$
- given a (syntactic) term t such that $\Gamma \vDash \Theta \mid \Delta \vdash t : A$, we interpret it as a linear term

$$[\![t]\!] : \mathrm{Tm}([\![\Gamma]\!] \vDash [\![\Theta]\!] \mid [\![\Delta]\!] \vdash [\![A]\!])$$

- given two terms u and v such that $\Gamma \vDash \Theta \mid \Sigma \mid \Delta \mid \Xi \vdash u \mapsto v : A$, it will hold that $[\![u]\!] \mapsto [\![v]\!]$

- given contexts Γ, Θ and Δ such that $\Gamma, \Theta \vdash \Delta$ ctxt, and a type A such that $\Gamma, \Theta \vdash A$ type, it will hold that

$$[\![\Gamma \vDash \Theta, x : A \vdash \Delta]\!] = [\![\Gamma \vDash \Theta \vdash \Delta]\!]\Big\{\overline{\mathbf{p}}_{[\![\Gamma \vDash \Theta \vdash A]\!]}\Big\}$$

- given contexts Γ, Θ and Δ such that $\Gamma, \Theta \vdash \Delta$ ctxt, it will hold that

$$[\![\Gamma, \Theta \vDash \diamond \vdash \Delta]\!] = [\![\Gamma \vDash \Theta \vdash \Delta]\!]\Big\{\nu_{[\![\Gamma \vDash \Theta]\!]}\Big\}$$

We do so, in every case, by induction on the derivation.

For instance, the $\{\}$-I case is

$$\frac{\Gamma \vDash \Theta \mid \diamond \mid \Delta \mid \diamond \vdash t \mapsto a : A}{\Gamma \vDash \Theta \mid \Delta \vdash \text{loop } t : \{a\}_A}\,\{\}\text{-I}$$

By induction hypothesis, we have $[\![t]\!] : \mathrm{Tm}([\![\Gamma]\!] \vDash [\![\Theta]\!] \mid [\![\Delta]\!] \vdash [\![A]\!])$, and $[\![a]\!] : \mathrm{Tm}\Big([\![\Gamma]\!], [\![\Theta]\!] \vDash \langle\rangle \vdash [\![A]\!]\Big\{\nu_{[\![\Theta]\!]}\Big\}\Big)$ such that $[\![t]\!] \mapsto [\![a]\!]$. We therefore have

$$[\![\text{loop } t]\!] = \mathrm{Loop}([\![t]\!], [\![a]\!])$$

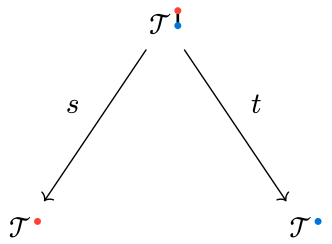
E Summary of Hanaba model

As the definition of a Hanaba model is quite large, we have summed up here all the *data* in its definition (that is, we do enumerate here the properties that we also assume on this structure).

F Chain model

The Hanaba syntax and semantics has been thought of by inspection of a concrete model which has been developed during the internship, the *chain model*. This model is the subject of the current section: in addition to exhibit its definition, we'll see that it is indeed a Hanaba model.

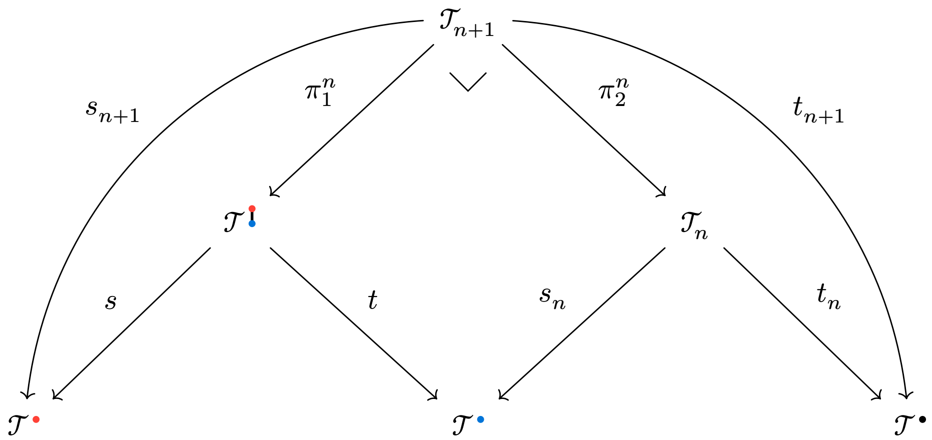
Assume we are given a double category



where t is a fibration, with multiplication $\circledast$ and unit e.

For $n : \mathbb{N}$, we will define a categorical span $\mathcal{T}^{\bullet} \xleftarrow{t_n} \mathcal{T}_n \xrightarrow{s_n} \mathcal{T}^{\bullet}$ by induction over n:

- $\mathcal{T}_0 := \mathcal{T}^{\bullet}$, with $t_0 = s_0 = \mathrm{id}_{\mathcal{T}^{\bullet}}$
- for $n : \mathbb{N}$, assume $\mathcal{T}^{\bullet} \xleftarrow{t_n} \mathcal{T}_n \xrightarrow{s_n} \mathcal{T}^{\bullet}$ given, consider the following pullback



Note that, for every $n : \mathbb{N}$, t_n is a fibration, by Proposition A.2.1 and Proposition A.2.2.

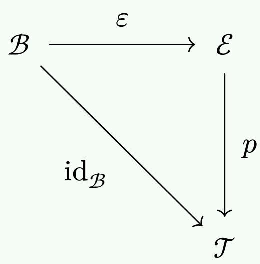
F.1 Chain-like structures

This subsection contains a definition and some properties of *chain-like structures*, which are artifacts for building the chain model. They are an abstract version of a chain category, and are used to prove several lemmas at once, without repeating oneself. Some intuition on them will probably emerge *after* having read the rest of the section, so in first reading, we suggest skipping this subsection.

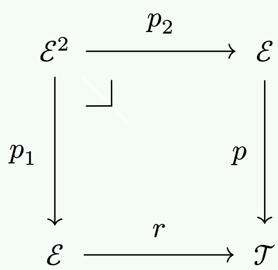
Definition F.1.1 (Chain-like structure)

A *chain-like structure* $C = (p, q, \varepsilon, r, \mu, s)$ is the data of

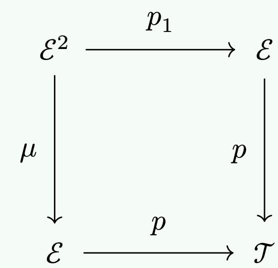
- a fibration $p : \mathcal{E} \to \mathcal{T}$
- a functor $q : \mathcal{C} \to \mathcal{T}$
- a section of p $\varepsilon : \mathcal{T} \to \mathcal{E}$, that is, such that the following diagram commutes



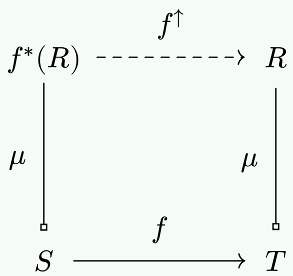
- a functor $r : \mathcal{E} \to \mathcal{T}$
- a functor $\mu : \mathcal{E}^2 \to \mathcal{E}$ where $\mathcal{E}^2$ is the following pullback



such that the following diagram commutes



and for any $f : S \to T$ in $\mathcal{E}$, and any $R : \mathcal{E}^2$ such that $\mu(R) = T$, there exists a unique object $f^*(R)$ in $\mathcal{E}^2$ and a unique morphism $f^\uparrow : f^*(R) \to R$ such that the following diagram commutes

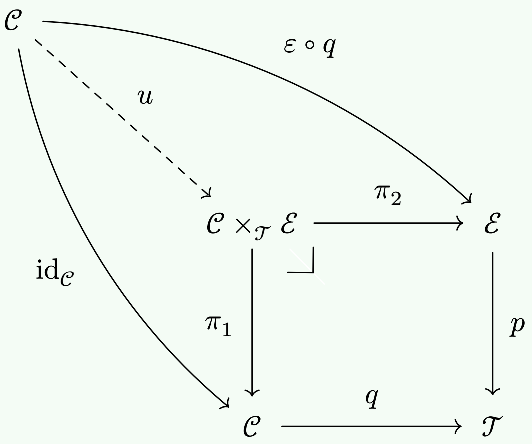


♣

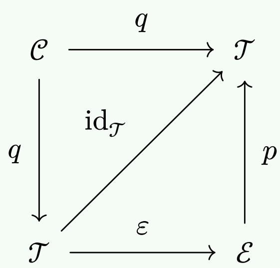
For the rest of the section, assume we are given a chain-like structure.

Definition F.1.2 (Unit of a chain-like structure)

We define the unit of the chain-like structure $u_C : \mathcal{C} \to \mathcal{C} \times_{\mathcal{T}} \mathcal{E}$ as the unique functor making the following diagram commute



This functor is well-defined because the following diagram commutes



Indeed, the right-most triangle commutes by definition of ε.

♣

Lemma F.1.3

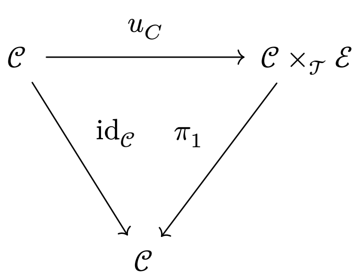
u_C is a fibration morphism

$$u_C : \mathrm{id}_{\mathcal{C}} \to \pi_1$$

Note that $\mathrm{id}_{\mathcal{C}}$ is a fibration by Proposition A.2.3.

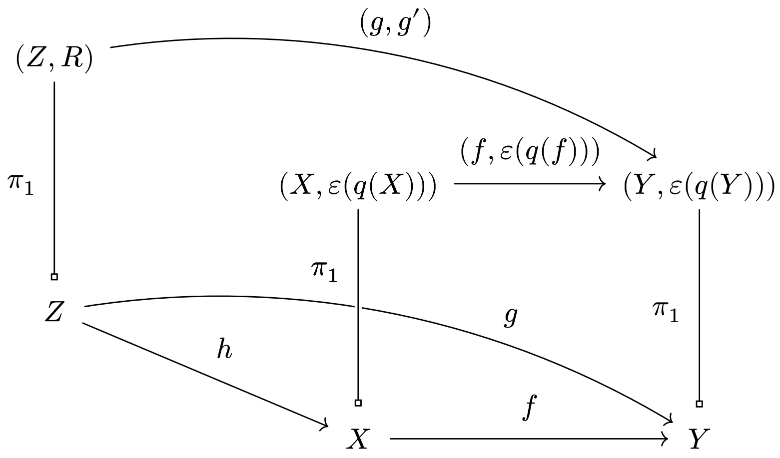
♡

Proof. The following diagram commutes by definition of u_C:



We furthermore need to check that u_C preserves cartesian morphisms, that is, since every morphism is cartesian with respect to the identity fibration, that u_C sends any morphism on a cartesian one.

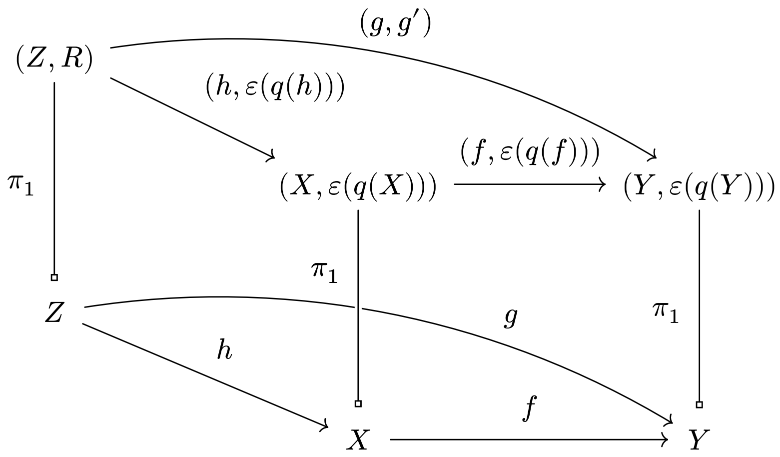
Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Let us show that $u_C(f)$ is cartesian too. Consider $(g, g') : (Z, R) \to (Y, \varepsilon(q(Y)))$ and $h : Z \to X$ making the following diagram commutes



Note that $q(g) = p(g')$, so

$$\varepsilon(q(g)) = \varepsilon(p(g')) = g'$$

Hence the following diagram commutes

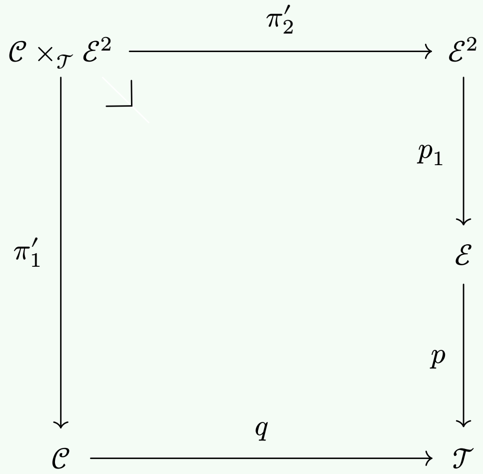


Furthermore, suppose that we have another h' such that (h, h') also makes the diagram commute. We would have $q(h) = p(h')$, and so by the same reasoning as before, $h' = \varepsilon(q(h))$, so h' is indeed unique.

Hence $u_C(f) = (f, \varepsilon(q(f)))$ is cartesian. □

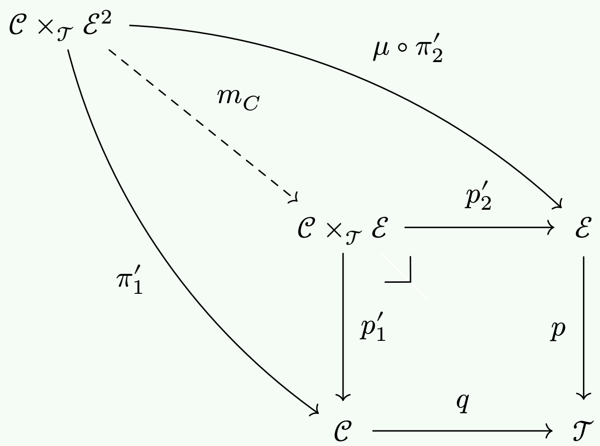
Definition F.1.4 (Multiplication of a chain-like structure)

Consider $\mathcal{C} \times_{\mathcal{T}} \mathcal{E}^2$ the following pullback

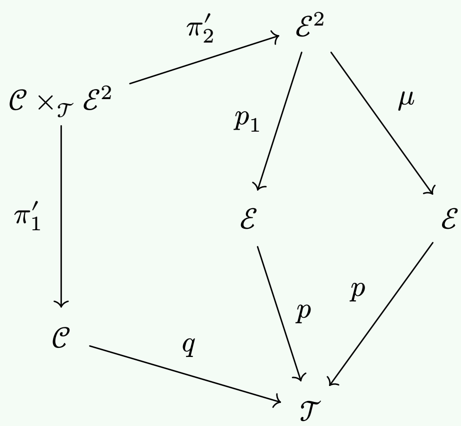


Since p_1 is the pullback of p by r, and p is a fibration, by Proposition A.2.1, p_1 is a fibration. Furthermore by Proposition A.2.2 $p \circ p_1$ is a fibration too, so by Proposition A.2.1, π'_1 is also a fibration.

Let $m_C : \mathcal{C} \times_{\mathcal{T}} \mathcal{E}^2$, the multiplication of C, be defined as the unique morphism making the following diagram commute.



This is well-defined as the following outer diagram commutes

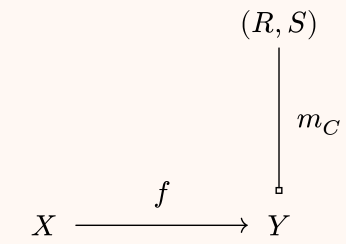


Indeed, the inner diagram on the right commutes by definition of μ, and the one on the left by definition of the pullback $\mathcal{C} \times_{\mathcal{T}} \mathcal{E}^2$.

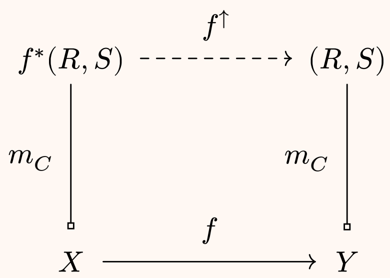
♣

Lemma F.1.5

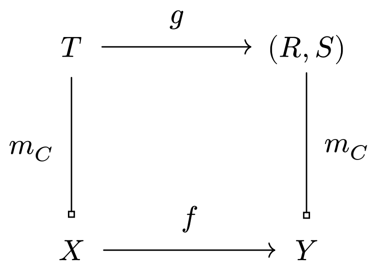
For any diagram of the form



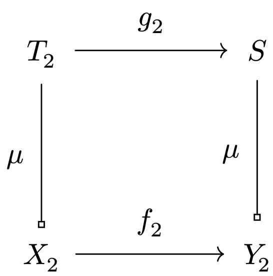
there exists a unique $f^*(R, S)$ in $\mathcal{C} \times_{\mathcal{T}} \mathcal{E}^2$ and a unique morphism $f^\uparrow : f^*(R, S) \to (R, S)$ making the following diagram commute



Proof. Suppose we have a $g : T \to (R, S)$ such that the following diagram commute



then in particular we have $g_1 = f_1$, and g_2 makes the following diagram commute

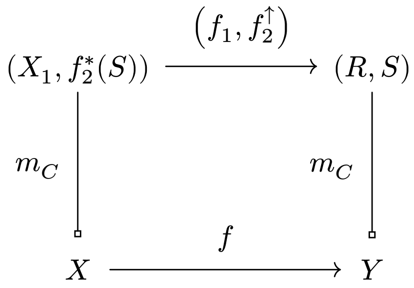


by assumption on μ, $g_2 = f_2^\uparrow$, so $g = \left(f_1, f_2^\uparrow\right)$ is uniquely determined.

Let us now prove existence, by showing that $\left(f_1, f_2^\uparrow\right)$ is a valid candidate. First of all, it is indeed a morphism in $\mathcal{C} \times_{\mathcal{T}} \mathcal{E}^2$:

$$\begin{aligned} p\left(p_1\left(f_2^\uparrow\right)\right) &= p\left(\mu\left(f_2^\uparrow\right)\right) && \text{by assumption on } \mu \\ &= p(f_2) && \text{by definition of } f_2^\uparrow \\ &= q(f_1) && \text{because } (f_1, f_2) \text{ is in } \mathcal{C} \times_{\mathcal{T}} \mathcal{E} \end{aligned}$$

Furthermore, the following diagram commutes



□

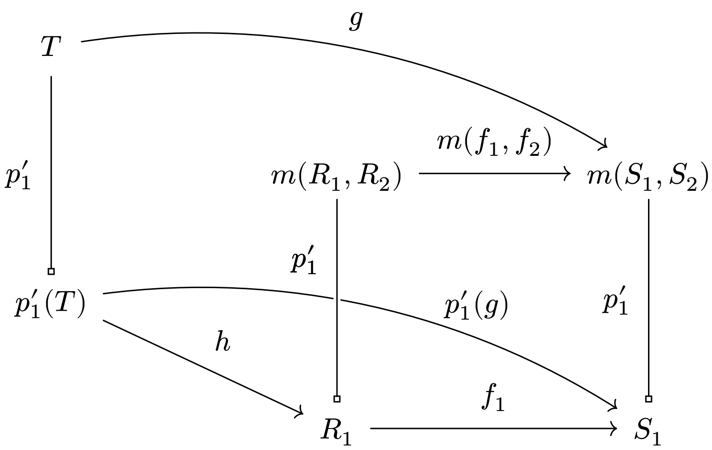
Lemma F.1.6

The multiplication m_C is a fibration morphism

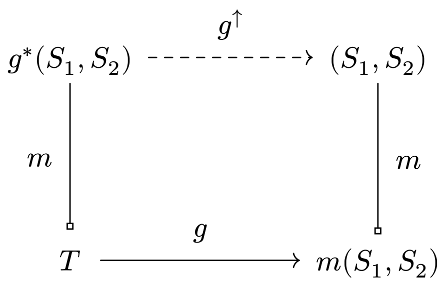
$$m_C : \pi_1' \to p_1'$$

Note that p_1' is a fibration as it is the pullback of p, by Proposition A.2.1.

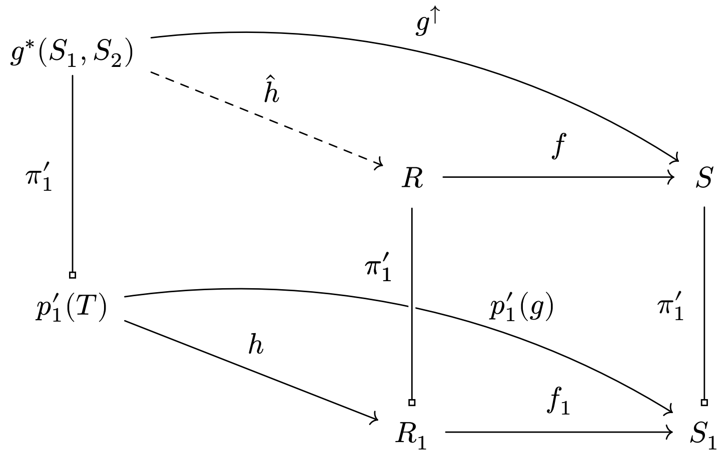
Proof. Consider a π_1'-cartesian morphism $f : (R_1, R_2) \to (S_1, S_2)$ in $\mathcal{C} \times_{\mathcal{T}} \mathcal{E}^2$. Let us show that $m(f_1, f_2)$ is p_1'-cartesian. Let $g : T \to m(S_1, S_2)$ and $h : p_1'(T) \to p_1'(R)$ making the following diagram commute



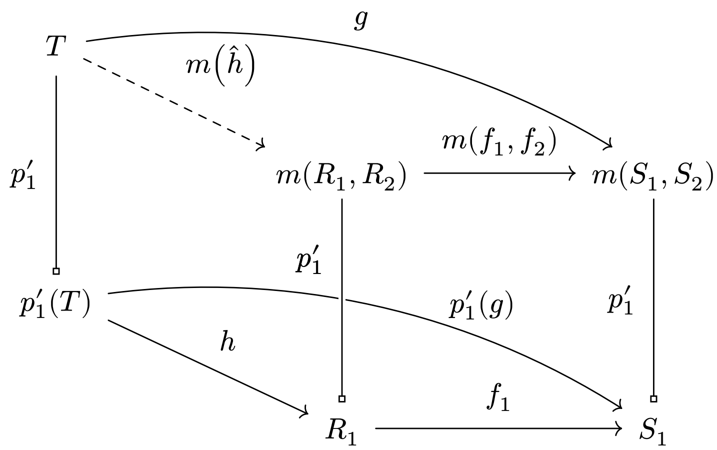
By Lemma F.1.5, there exists a unique $g^{\uparrow} : g^*(S_1, S_1) \to (S_1, S_2)$ making the following diagram commute



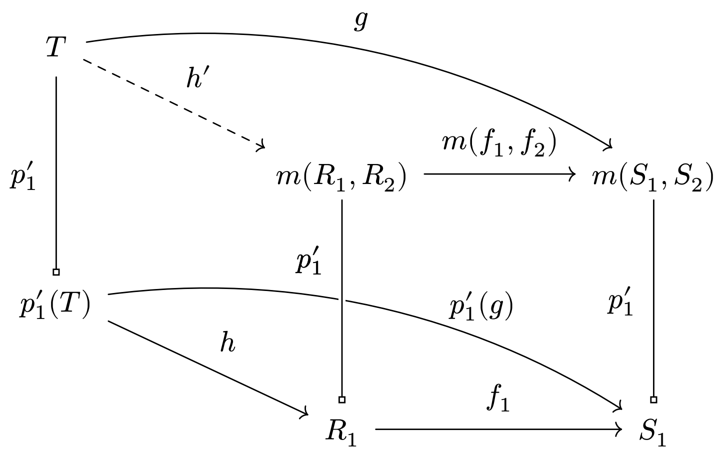
By cartesianity of f, there exists a unique $\hat{h}$ making the following diagram commute



So in particular, the following diagram commutes



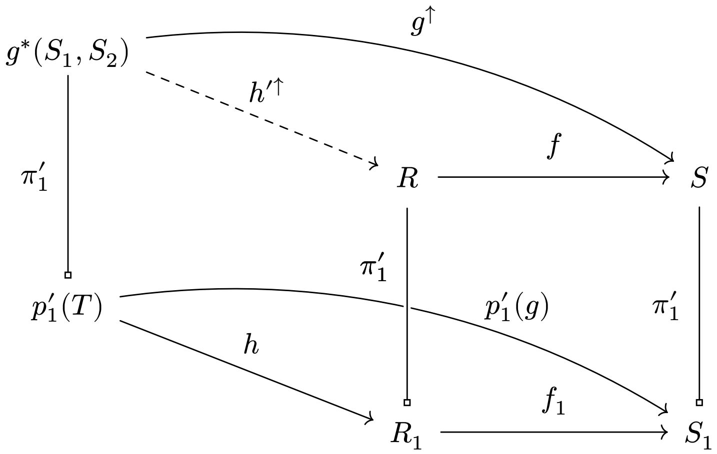
Let us prove that $m(\hat{h})$ is unique. Consider a $h' : T \to m(R_1, R_2)$ making the following commute



By Lemma F.1.5, we can lift h' and it must agree with the lifting of g:

$$g^\uparrow = f \circ h'^\uparrow$$

Hence, the following diagram commutes



Hence $h'^\uparrow$ satisfies the same universal property as $\hat{h}$, so we have

$$h'^\uparrow = \hat{h}$$

and so

$$m\left(\hat{h}\right) = m(h'^\uparrow) = h'$$

□

F.2 The global context category

Definition F.2.1 (Context)

A *context* is a pair $(|\Gamma|, \Gamma)$ with $|\Gamma| : \mathbb{N}$ and $\Gamma : \mathcal{T}_{|\Gamma|}$ a chain. We will often denote such a context by Γ.

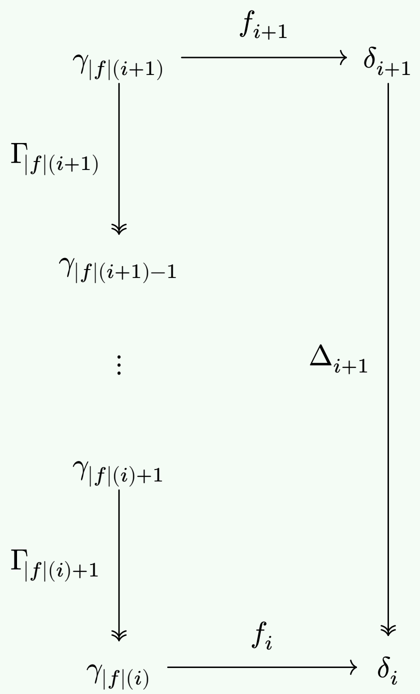
♣

Definition F.2.2 (Substitution)

A *substitution* f from Γ to Δ is a monotonous map

$$|f| : \{0, ..., |\Delta|\} \to \{0, ..., |\Gamma|\}$$

such that $|f|(|\Delta|)$, and morphisms $f_i : \gamma_{|f|(i)} \to \delta_i$, such that for every i, the following diagram "commutes"



that is, there is a morphism

$$f^{\bullet}_{i+1} : \Gamma_{|f|(i+1)} \otimes \ldots \otimes \Gamma_{|f|(i)+1} \longrightarrow \Delta_{i+1}$$

such that

$$s(f^{\bullet}_{i+1}) = f_{i+1}$$
$$t(f^{\bullet}_{i+1}) = f_i$$

Note that the $f^{\bullet}_i$ are part of the data of f.

Let Γ, Δ and Θ be three contexts, and $f : \Gamma \to \Delta$ and $g : \Delta t \to \Theta$ be two substitutions. We define their composition as follows:

$$|g \circ f| = |f| \circ |g|$$
$$(g \circ f)_i = g_i \circ f_{|g|(i)}$$

Furthermore, each square of the composition commutes, and the composition is commutative, because we are composing 2-cells as in an fc-multicategory [18], [19].

Definition F.2.3 (Context category)

Let us write $\mathcal{B}$ the *context category*, that is, the category whose objects are contexts, and with morphisms substitutions.

Definition F.2.4 (Flat context category)

Let $\mathcal{B}^{\bullet}$ be the *flat context category*, the subcategory of $\mathcal{B}$ with the same objects, and morphisms f are morphisms in $\mathcal{B}$ such that $|f|(0) = 0$.

Definition F.2.5 (Chain target functor)

Let $\hat{t} : \mathcal{B}^\bullet \to \mathcal{T}^\bullet$ be the *chain target functor*, defined by

$$\begin{aligned} \hat{t} : \mathcal{B}^\bullet &\longrightarrow \mathcal{T}^\bullet \\ \Gamma &\longmapsto \gamma_0 \\ f &\longmapsto f_0 \end{aligned}$$

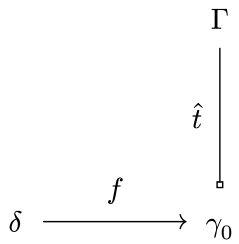
♣

Lemma F.2.6

$\hat{t}$ is a fibration.

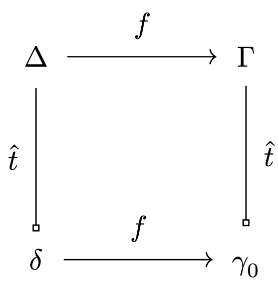
♡

Proof. Let Γ be a context, $\delta : \mathcal{T}^\bullet$ and $f : \delta \to \gamma_0$.



Let us lift f by induction on $|\Gamma|$.

If $|\Gamma| = 0$, then $[f]^{\hat{t}}_{\Gamma}$ is just f, with $\Delta = \delta$

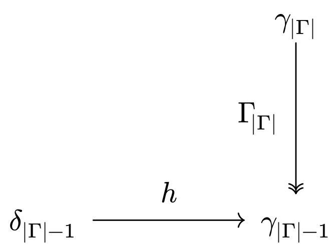


It is cartesian. Indeed, assume we have a $g : \Theta \to \Delta$. Then $|\Theta| = |g|(|\Delta|) = |g|(0) = 0$, so if we have an $h : \theta_0 \to \gamma_0$ such that $\pi(g) = f \circ h$, then in fact $g = f \circ h$ so h is its own lift.

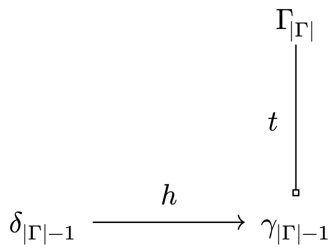
Otherwise, assume that we have built a cartesian $[f]^{\hat{t}}_{\Delta|_n} : \Delta \to \Gamma|_{|\Gamma|-1}$ and let us extend it to

$$[f]^{\hat{t}}_{\Gamma} : \Delta^\uparrow \to \Gamma$$

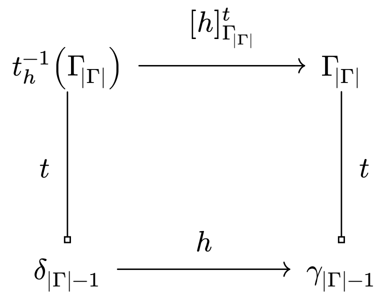
We are in the following situation



with $h = \left([f]^{\hat{t}}_{|\Gamma|-1}\right)_{|\Gamma|-1}$, so in particular



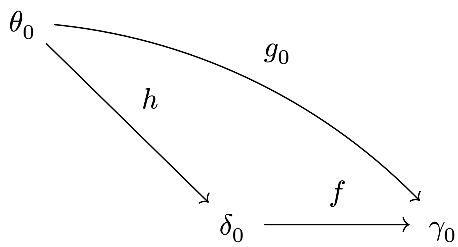
Because t is a fibration, we can lift h as follows



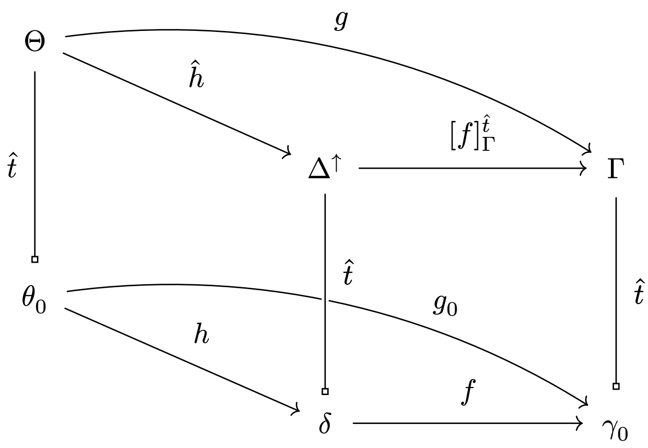
so we let

$$\begin{aligned} \delta_{|\Gamma|} &= s\left(t_h^{-1}\left(\Gamma_{|\Gamma|}\right)\right) \\ \Delta_{|\Gamma|} &= t_h^{-1}\left(\Gamma_{|\Gamma|}\right) \\ \left([f]^{\hat{t}}_{\Gamma}\right)_{|\Gamma|} &= s\left([h]^t_{\Gamma_{|\Gamma|}}\right) \\ \left([f]^{\hat{t}}_{\Gamma}\right)^{\bullet}_{|\Gamma|} &= [h]^t_{\Gamma_{|\Gamma|}} \\ |[f]^{\hat{t}}_{\Gamma}|(|\Gamma|) &= |\Gamma| \end{aligned}$$

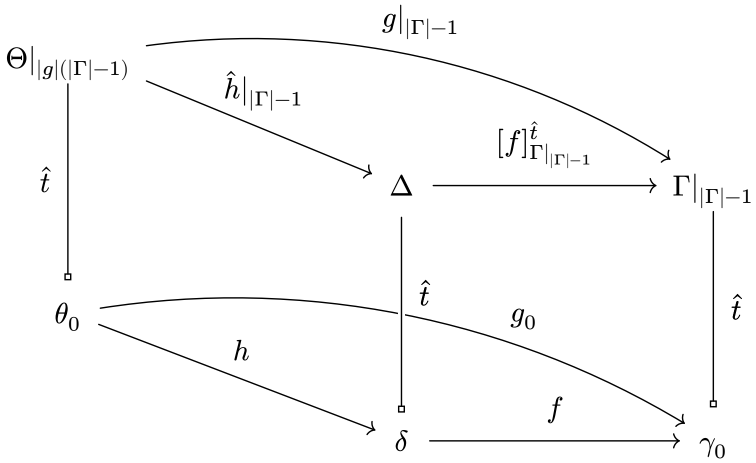
Let us show that this extended morphism is still cartesian. Consider a context Θ, a $g : \Theta \to \Gamma$ and a $h : \theta_0 \to \delta_0$ such that the following diagram commutes



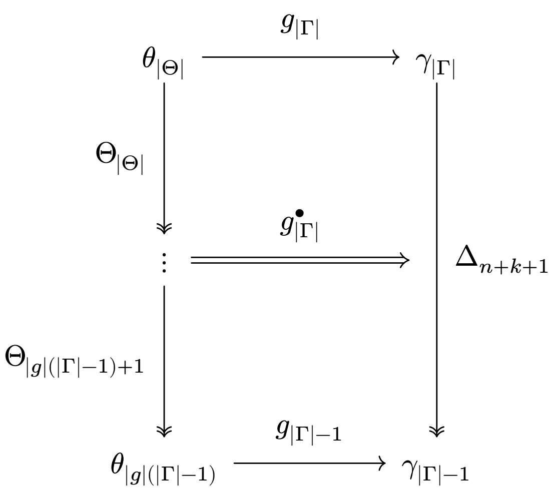
Consider a lifting $\hat{h}$ of h making the following diagram commmute.



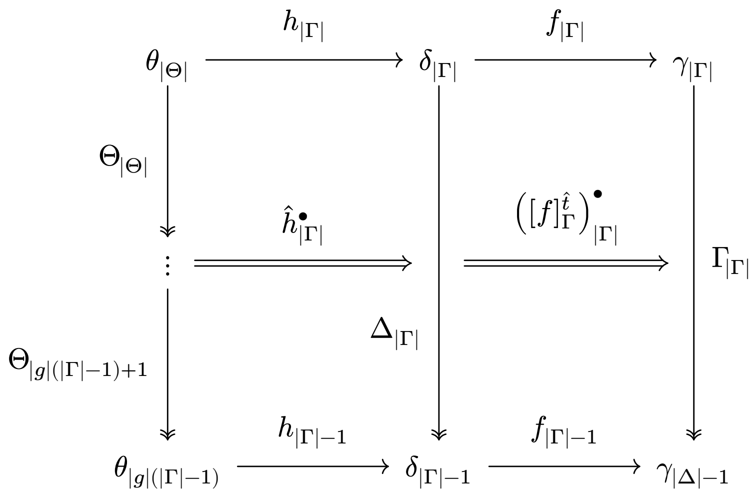
note that $\hat{h}$ is uniquely determined by the data of $\hat{h}|_{|\Gamma|-1}$ and $\hat{h}^{\bullet}_{|\Gamma|}$. But $\hat{h}|_{|\Gamma|-1}$ makes the following diagram commute



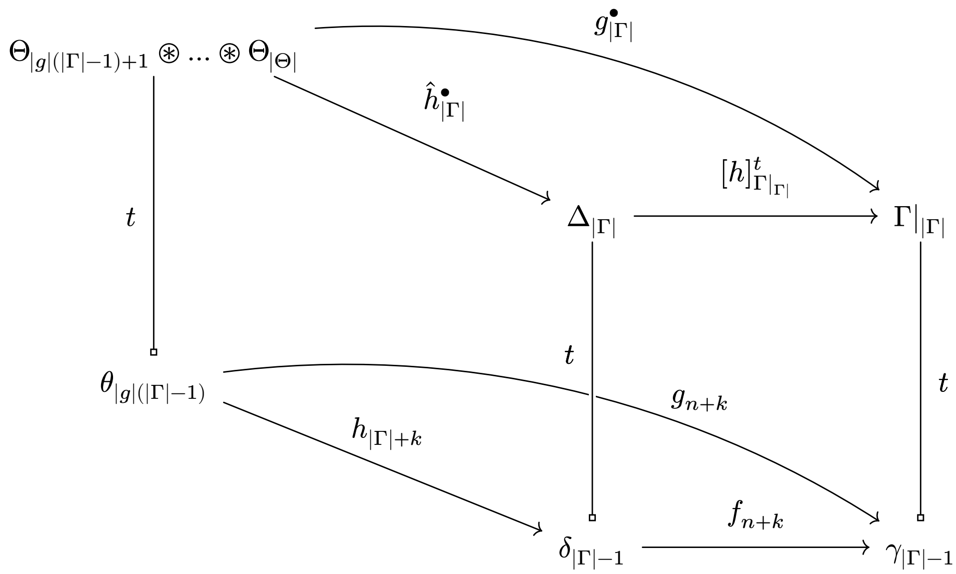
so it uniquely exists, by cartesianity of $[f]^{\hat{t}}_{\Gamma|_{|\Gamma|-1}}$. Furthermore, the commutation of the previous diagram also implies the following two pastings are the same:



and



which is equivalent to the commutation of the following diagram



hence, $\hat{h}^{\bullet}_{|\Gamma|}$ uniquely exists. Thus, $\hat{h}$ uniquely exists, showing the cartesianity of $[f]^{\hat{t}}_{\Gamma}$. □

Definition F.2.7

Let $\varepsilon : \mathcal{T}^{\bullet} \to \mathcal{B}^{\bullet}$ be the section of $\hat{t}$ defined by $\varepsilon(\gamma) = \gamma$. ♣

Definition F.2.8

Let $\mathcal{B}^{\bullet}_{\star}$ be the pointed flat context category, whose elements are pointed flat contexts $(\Gamma, |\Gamma|)$, and whose morphisms f are pointed substitutions such that $|f|(0) = 0$. ♣

Lemma F.2.9

The following diagram is a pullback

$$
\begin{array}{ccc}
\mathcal{B}^\bullet_\star & \longrightarrow & \mathcal{B}^\bullet \\
\big\downarrow & \lrcorner & \big\downarrow \hat{t} \\
\mathcal{B}^\bullet & \xrightarrow{\hat{s}} & \mathcal{T}^\bullet
\end{array}
$$

♡

Proof. Immediate. □

Definition F.2.10

Let $\mu : \mathcal{B}^\bullet_\star \to \mathcal{B}^\bullet$ be defined by

$$
\begin{aligned}
\mu : \quad \mathcal{B}^\bullet_\star &\longrightarrow \mathcal{B}^\bullet \\
(\Gamma, n) &\longmapsto \Gamma \\
f &\longmapsto f
\end{aligned}
$$

♣

Lemma F.2.11

This defines a chain-like structure.

♡

F.3 The pointed context category

Definition F.3.1 (Pointed context)

A *pointed context* is a pair (Γ, n) with $\Gamma : \mathcal{B}$ and $n : \{0, ..., |\Gamma|\}$.

♣

Note that this definition justifies the name *pointed context*: indeed, a *pointed context* is a context with a distinguished degree. As we will see, pointed substitutions are, as we expect, substitutions that preserve it.

Definition F.3.2 (Pointed substitutions)

A *pointed substitution* fro (Γ, n) to (Δ, m) is a substitution $f : \Gamma \to \Delta$ such that $|f|(m) = n$.

♣

Definition F.3.3 (Pointed context category)

Let $\mathcal{B}_*$ be the *pointed context category*, the category with objects pointed contexts, and with morphisms pointed substitutions.

♣

Definition F.3.4 (Global projection)

The *global projection* π is the functor

$$
\begin{aligned}
\pi : \quad \mathcal{B}_* &\longrightarrow \mathcal{B} \\
(\Gamma, n) &\longmapsto \Gamma|_n
\end{aligned}
$$

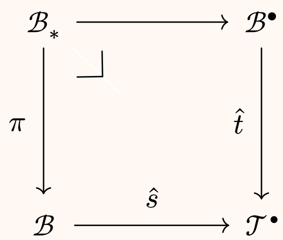
where

$$\left(\gamma_{|\Gamma|} \twoheadrightarrow \cdots \twoheadrightarrow \gamma_0\right)|_n = \gamma_n \twoheadrightarrow \cdots \twoheadrightarrow \gamma_0$$

and whose action on morphisms is clear. ♣

Lemma F.3.5

The following diagram is a pullback



♡

Proof. Immediate. □

Corollary F.3.5.1

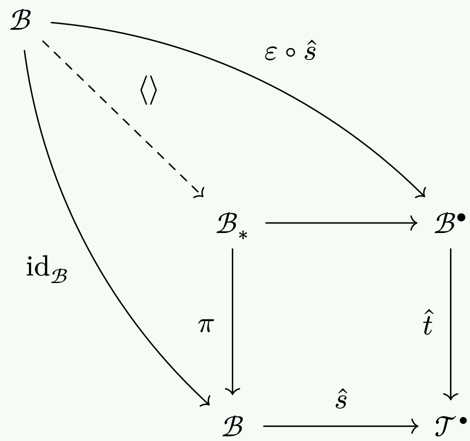
π is a Grothendieck fibration. ♡

Proof. By Proposition A.2.1. □

F.4 Fibered double category structure

Definition F.4.1 (Empty context)

Let $\langle\rangle : \mathcal{B} \to \mathcal{B}_*$ be the empty context functor



♣

Definition F.4.2 (Forgetful functor)

The *forgetful functor* is the functor

$$\begin{aligned} F : \quad \mathcal{B}_* &\longrightarrow \mathcal{B} \\ (\Gamma, n) &\longmapsto \Gamma \\ f &\longmapsto f \end{aligned}$$

Definition F.4.3 (Context multiplication)

Let us define the functor $-, - : \mathcal{B}_* \times_{\mathcal{B}} \mathcal{B}_* \to \mathcal{B}$ as follows.

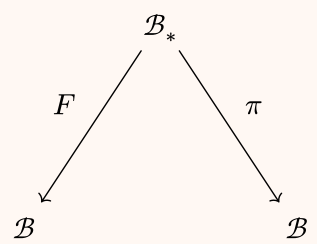
Let (Γ, n) and (Δ, m) be in $\mathcal{B}$ such that $\pi(\Gamma, n) = F(\Delta, m)$, that is, $\Delta = \Gamma|_n$. Put

$$(\Delta, m), (\Gamma, n) = (\Gamma, m)$$

For any f and g such that $\pi(f) = F(g)$, $f, g = f$.

Theorem F.4.4

The following



forms a fibered double category, with unit $\langle\rangle$ and multiplication $-, -$.

Proof. See Appendix H. □

F.5 The type category

Definition F.5.1 (Chain source functor)

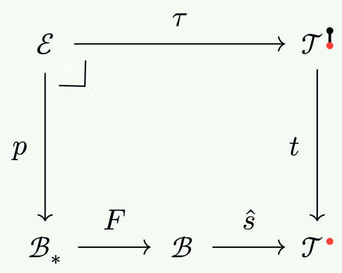
The *chain source functor* is the functor

$$\begin{aligned} \hat{s} : \mathcal{B} &\longrightarrow \mathcal{T}^{\bullet} \\ \Gamma &\longmapsto \gamma_{|\Gamma|} \end{aligned}$$

with obvious action on morphisms.

Definition F.5.2 (Type category)

The *type category* $\mathcal{E}$ is defined as the following pullback



Definition F.5.3

Let $q : \mathcal{E} \to \mathcal{B}_*$ be defined by associating to the type $((\Gamma, n), A)$ the context $(\Gamma.A, n)$ where $\Gamma.A$ is

$$\begin{array}{c} s(A) \\ \Big\downarrow{\scriptstyle A} \\ \gamma_{|\Gamma|} \\ \Big\downarrow{\scriptstyle \Gamma_{|\Gamma|}} \\ \vdots \\ \Big\downarrow{\scriptstyle \Gamma_1} \\ \gamma_0 \end{array}$$

and similarly on morphisms.

Definition F.5.4

Let us define a unit $1 : \mathcal{B}_* \to \mathcal{E}$ by

$$1 = \langle \mathrm{id}_{\mathcal{B}_*}, U \circ \hat{s} \circ F \rangle$$

Definition F.5.5

Let us define a multiplication $\Sigma : \mathcal{E} \times_{\mathcal{B}_*} \mathcal{E} \to \mathcal{E}$ by

$$\Sigma(\Theta, A, B) := (\Theta, A \circledast B)$$

and similarly for morphisms.

Theorem F.5.6

The following

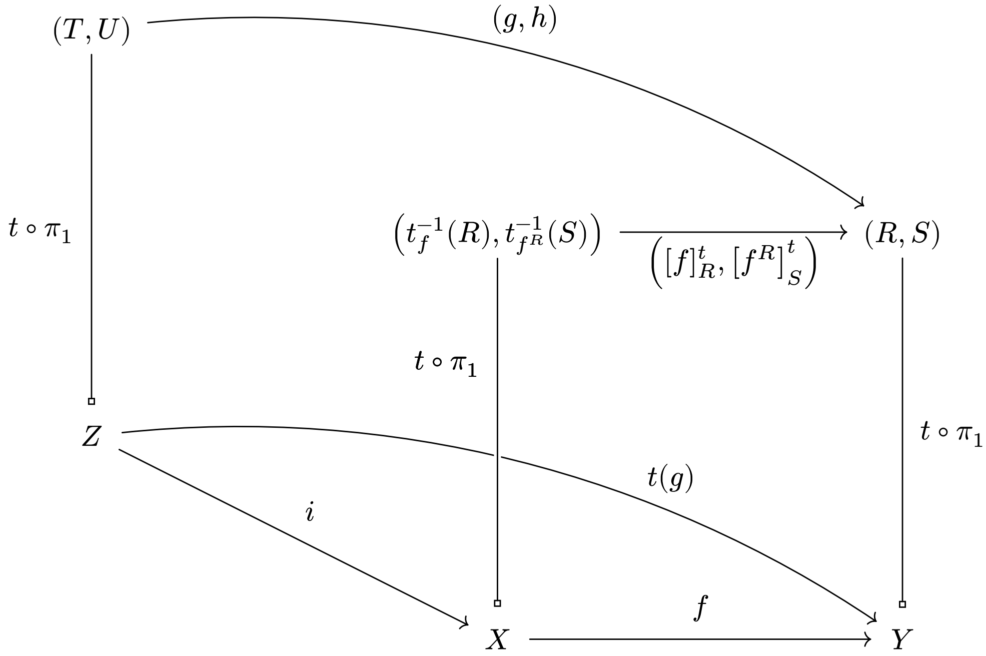
$$\mathcal{B}_* \xleftarrow{\;q\;} \mathcal{E} \xrightarrow{\;p\;} \mathcal{B}_*$$

is a fibered double category.

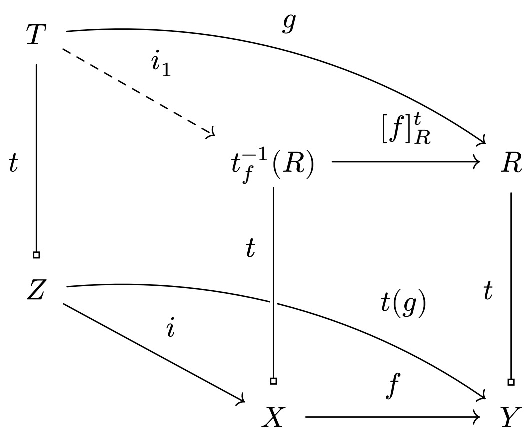
Proof. See Appendix I ☐

G Proof of Proposition 3.2.5

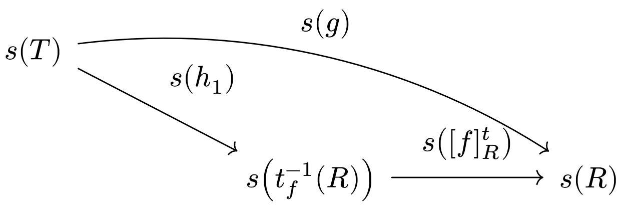
Proof. Let $f : X \to Y$ be a morphism in $\mathcal{E}^{\bullet}$, $R : t_Y^{-1}$ and $S : t_{s(R)}^{-1}$. We want to show that $\left([f]_R^t, \left[f^R\right]_S^t\right)$ is a cartesian morphism. Consider the following situation, with $Z : \mathcal{E}^{\bullet}$, $T : t_Z^{-1}$, $U : t_{s(T)}^{-1}$, $g : T \to R$ and $h : U \to S$ two morphisms in $\mathcal{E}^{\bullet|}$ such that $t(h) = s(g)$. Let $i : T \to X$ such that the following diagram commutes



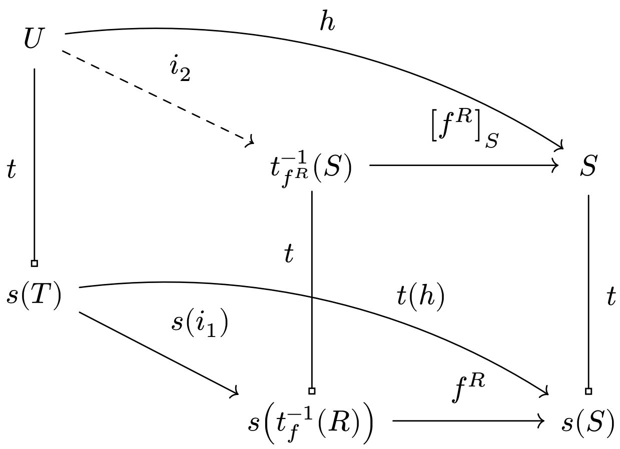
In particular, by cartesianity of $[f]_R^t$, there exists a unique $i_1 : T \to t_f^{-1}(R)$ making the following diagram commute



Furthermore, if we take the image of the top diagram by s, we have that the following diagram commutes



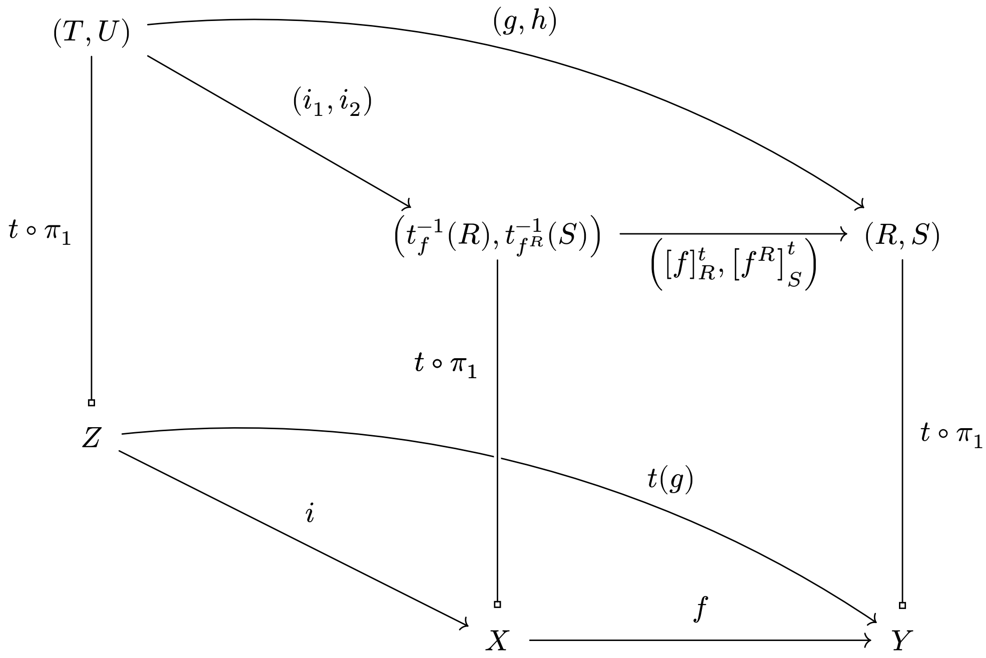
Since $\left[f^R\right]_S$ is cartesian, and because $t(h) = s(g)$ and $f^R = s([f]_R)$, we have that there exists a unique $i_2 : U \to t_{f^R}^{-1}(S)$ making the following diagram commute



Hence, we have the following:

- $t(i_2) = s(i_1)$, hence (i_1, i_2) is a morphism in $\mathcal{E}^{\vdots}$;
- $(t \circ \pi_1)(i_1, i_2) = t(i_1) = i$, so (h_1, h_2) lives above h;
- $(g, h) = \Big([f]^t_R, [f^R]^t_S\Big) \circ (i_1, i_2)$ by construction.

So indeed, we have found a morphism that makes the following diagram commute.



Now, let's show the unicity. Suppose we have some $(j_1, j_2) : (T, U) \to \big(t_f^{-1}(R), t_{f^R}^{-1}(S)\big)$ making the latter diagram commute. Then, by projecting with t, j_1 satisfies the same universal property as i_1, so $j_1 = i_1$, and, using that, and by projecting with s, j_2 satisfies the same universaly property as i_2, so $j_2 = i_2$, and so

$$(j_1, j_2) = (i_1, i_2)$$

□

H Proof of Theorem F.4.4

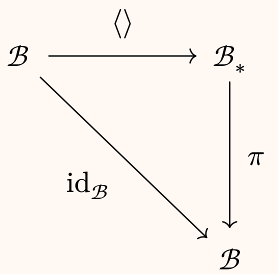
Lemma H.1

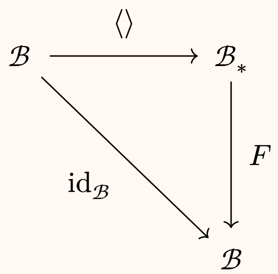
The global projection π is a fibration.

Proof. By Proposition A.2.1. □

Lemma H.2

The following diagrams commute





Proof. Immediate. □

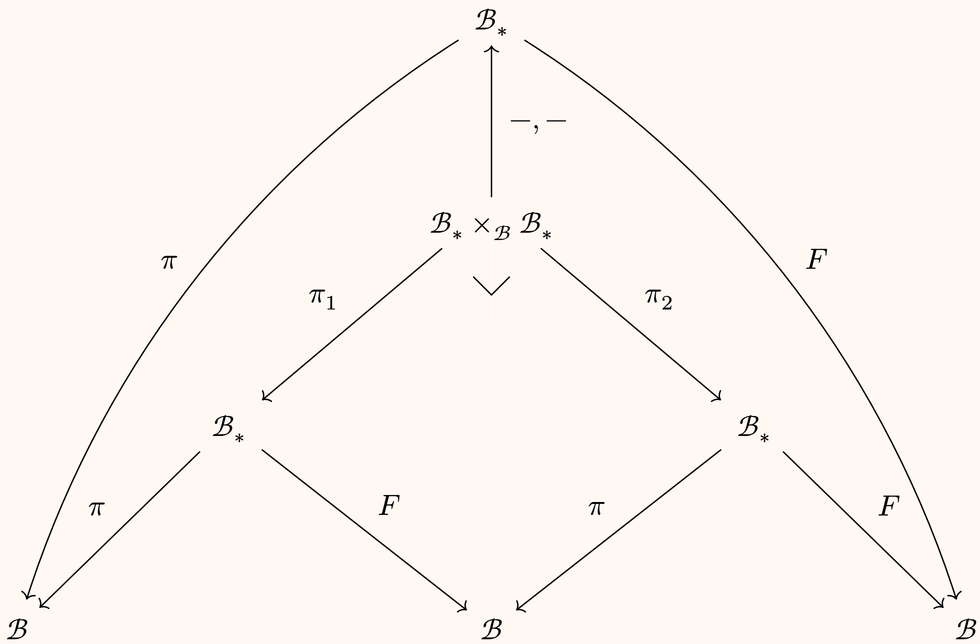
Lemma H.3

$\langle\rangle : \mathrm{id}_\mathcal{B} \to \pi$ is a fibration morphism.

Proof. By Lemma F.1.3. □

Lemma H.4

The following diagram commutes



Proof. Consider (Δ, n, m) in $\mathcal{B}_* \times_\mathcal{B} \mathcal{B}_*$. For the left triangle, we have

$$\begin{aligned}\pi((\Delta|_m, n), (\Delta, m)) &= \pi(\Delta, n) \\ &= \Delta|_n \\ &= \pi(\Delta|_m, n)\end{aligned}$$

For the right triangle, we have

$$\begin{aligned}F((\Delta|_m, n), (\Delta, m)) &= F(\Delta, n) \\ &= \Delta \\ &= F(\Delta, m)\end{aligned}$$

commutation for morphisms is just as easy. □

Lemma H.5

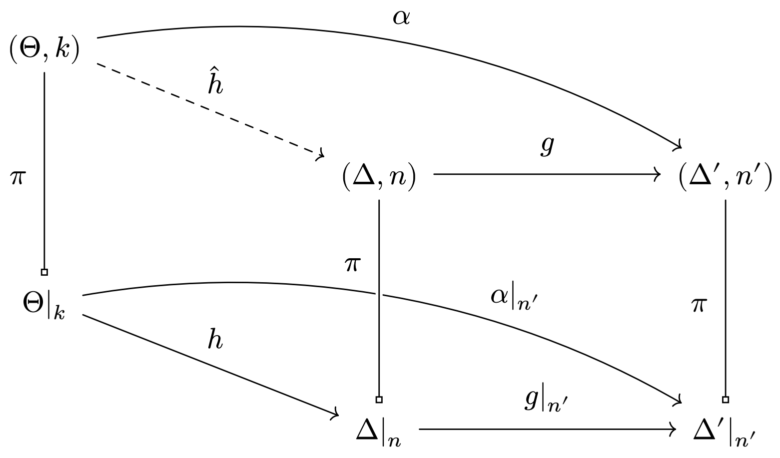
The multiplication is associative

Proof. Immediate. □

Lemma H.6

The multiplication is a fibration morphism $-,- : \pi \circ \pi_1 \to \pi$.

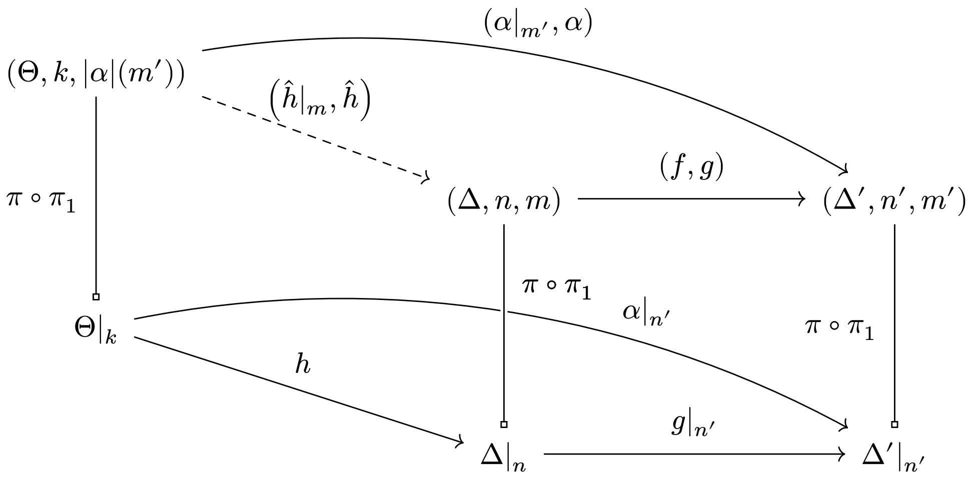
Proof. Consider a morphism $(f, g) : (\Delta, n, m) \to (\Delta', n', m')$ in $\mathcal{B}_* \times_{\mathcal{B}} \mathcal{B}_*$, that is, $F(f) = \pi(g)$, ie. $f = g|_{m'}$. Suppose (f, g) is cartesian. We want to show that $f, g = g$ is also π-cartesian. Suppose given $\alpha : (\Theta, k) \to (\Delta', n')$ and $h : \Theta|_k \to \Delta'|_{n'}$ such that the following diagram commutes, and consider an $\hat{h} : (\Delta, k) \to (\Delta, n)$ making the following diagram commute



Then we have

$$\begin{aligned}|\hat{h}|(m) &= |\hat{h}|(|g|(m')) \\ &= |\alpha|(m')\end{aligned}$$

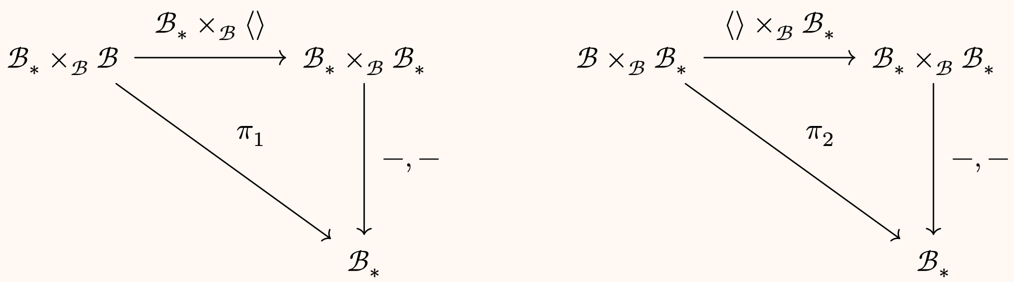
so such an $\hat{h}$ is exactly one making the following diagram commute



so it exists and is unique, by cartesianity of (f, g). □

Lemma H.7

The following two diagrams commute



Proof. Immediate. □

I Proof of Theorem F.5.6

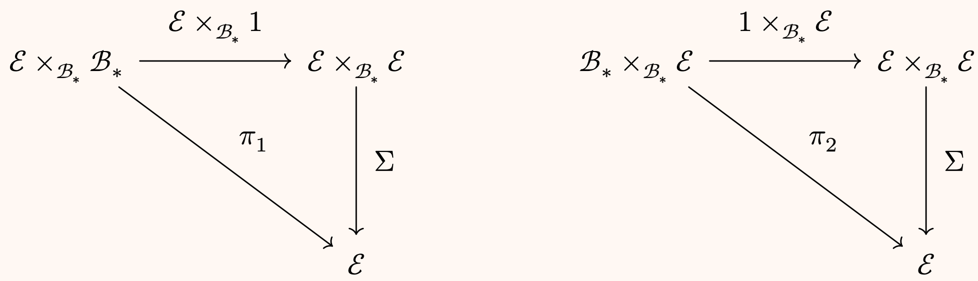
Lemma I.1

p is a fibration.

Proof. By Proposition A.2.1, because t is a fibration. □

Lemma I.2

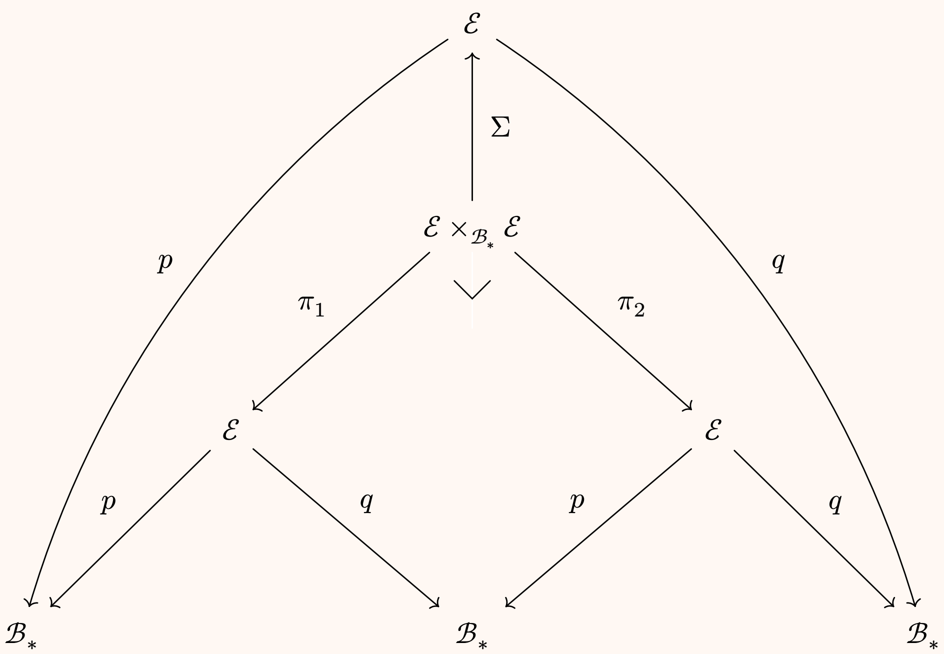
The following diagrams commute



Proof. Immediate. □

Lemma I.3

The following diagram commutes



Proof.

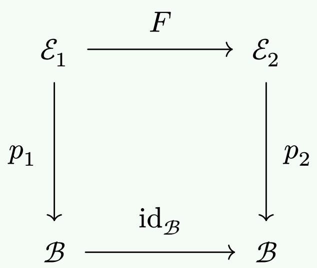
□

J Proof of Theorem A.3.1

Definition J.1 (Category of fibrations)

For a base category $\mathcal{B}$, define $\mathbf{Fib}_{\mathcal{B}}$ as the category of fibrations over $\mathcal{B}$, that is, whose objects are pairs $(\mathcal{E}, p)$ with $\mathcal{E}$ a category and $p : \mathcal{E} \to \mathcal{B}$ a fibration.

Given two fibrations $p_i : \mathcal{E}_i \to \mathcal{B}$ over $\mathcal{B}$ for $i = 1, 2$, a morphism of fibrations between p_1 and p_2 is a functor $F : \mathcal{E}_1 \to \mathcal{E}_2$ making the following diagram commute



and which preserves cartesianity of morphisms. ♣

Definition J.2 (Category of pseudofunctors)

For a given base category $\mathcal{B}$, define $\mathbf{Pfct}_{\mathcal{B}}$ as the category whose elements are contravariant pseudo-functors $\mathcal{P} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ in $\mathbf{Cat}$, that is,

- for each object $X : \mathcal{B}$, a category $\mathcal{P}_X$;
- for each morphism $f : X \to Y$ in $\mathcal{B}$, a functor $\mathcal{P}_f : \mathcal{P}_Y \to \mathcal{P}_X$;
- for each object $X : \mathcal{B}$, a natural isomorphism

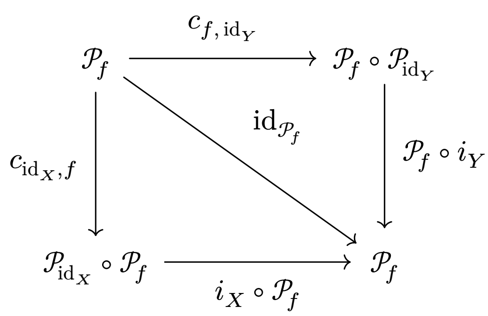
$$i_X : \mathcal{P}_{\mathrm{id}_X} \Longrightarrow \mathrm{id}_{\mathcal{P}_X}$$

  called the pseudo unit of $\mathcal{P}$ at X;
- for each morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{B}$, a natural isomorphism

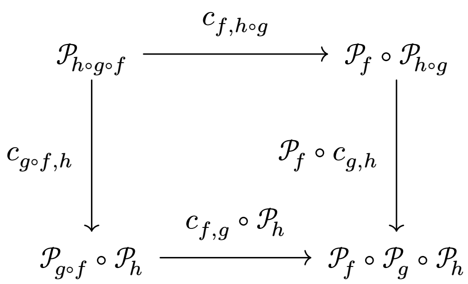
$$c_{f,g} : \mathcal{P}_{g \circ f} \Longrightarrow \mathcal{P}_f \circ \mathcal{P}_g$$

  called the pseudo composition law of $\mathcal{P}$ at f and g. ♣

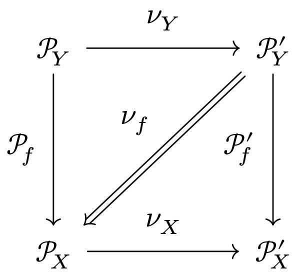
We additionally require the following coherence conditions: for $f : X \to Y$, the following diagram commutes



Furthermore, for $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, the following diagram commutes

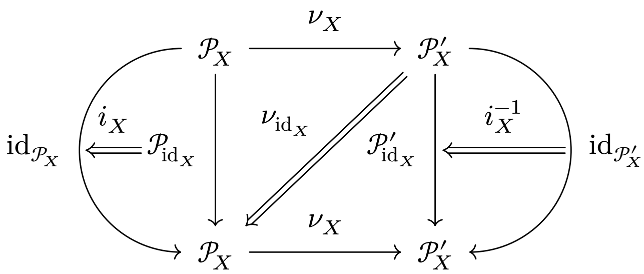


Given two pseudofunctors $\mathcal{P}$ and $\mathcal{P}'$, a morphism $\nu : \mathcal{P} \to \mathcal{P}'$ is a pseudonatural transformation between $\mathcal{P}$ and $\mathcal{P}'$, that is, for each point $X : \mathcal{B}$, a functor $\nu_X : \mathcal{P}_X \to \mathcal{P}'_X$ and, for each morphism $f : X \to Y$ in $\mathcal{B}$, a natural isomorphism



satisfying the following coherence conditions:

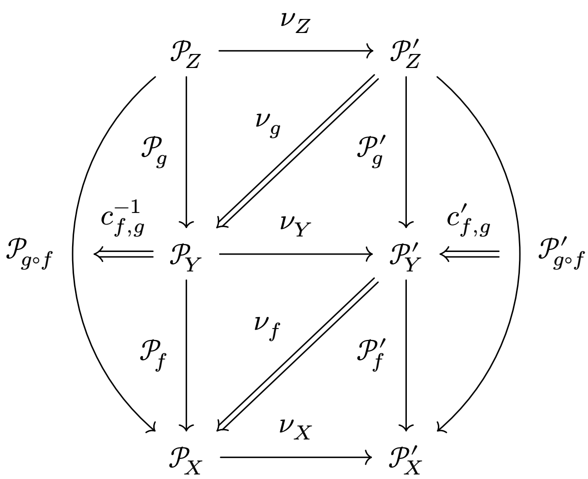
- for $X : \mathcal{B}$, the following pasting is ν_X



  that is,

$$(\nu_X \circ i_X) \circ \nu_{\mathrm{id}_X} \circ (i'^{-1}_X \circ \nu_X) = \mathrm{id}_{\nu_X}$$

- if $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $\mathcal{B}$, $\nu_{g\circ f}$ is obtained by pasting the squares (plus pseudo-composition)



that is,

$$\nu_{g\circ f} = \left(\nu_X \circ c^{-1}_{f,g}\right) \circ \left(\nu_f \circ \mathcal{P}_g\right) \circ \left(\mathcal{P}'_f \circ \nu_g\right) \circ \left(c'_{f,g} \circ \nu_Z\right)$$

We aim at proving that for a given base category $\mathcal{B}$, we have

$$\mathbf{Fib}_{\mathcal{B}} \cong \mathbf{Pfct}_{\mathcal{B}}$$

In order to do so, we will build in Appendix J.1 half of the equivalence, namely,

$$\Phi : \mathbf{Fib}_{\mathcal{B}} \to \mathbf{Pfct}_{\mathcal{B}}$$

and, in Appendix J.2, the other half of the equivalence, namely,

$$\Psi : \mathbf{Pfct}_{\mathcal{B}} \to \mathbf{Fib}_{\mathcal{B}}$$

In Appendix J.3, we will show that the two form the two halves of an equivalence, finishing the proof.

J.1 Fiber functor

Let's build

$$\Phi : \mathbf{Fib}_{\mathcal{B}} \to \mathbf{Pfct}_{\mathcal{B}}$$

J.1.1 Action of Φ on objects

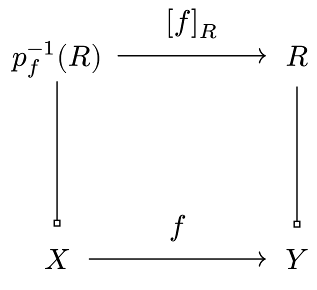
Assume we have a fibration p.

J.1.1.1 Definition of the fibre pseudo-functor

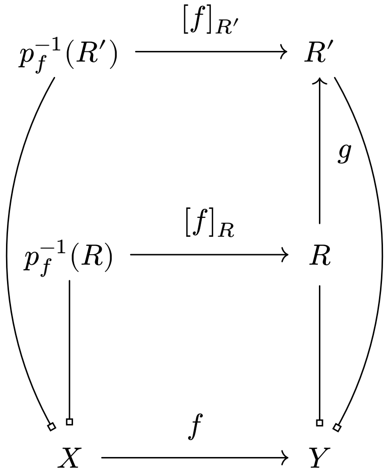
Let us build $p^{-1} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ a pseudo-functor. For $X : \mathcal{B}$,

$$p^{-1}_X = \{R : \mathcal{E} \mid R \sqsubset X\}$$
$$p^{-1}_X(S, R) = \{\alpha : S \to R \mid p(\alpha) = \mathrm{id}_X\}$$

Let $X, Y : \mathcal{B}$ and $f : X \to Y$. Let's define $p^{-1}_f : p^{-1}_Y \to p^{-1}_X$ by noticing that, for each $R : p^{-1}_Y$, by the fibration condition on p, there exists a cartesian morphism $[f]_R$



Furthermore, for $R, R' : p^{-1}_Y$ and $g : R \to R'$, we have the following diagram

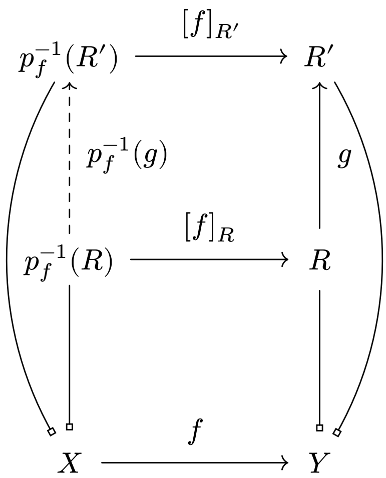


By cartesianity of $\iota_{R'}$, there exists a unique $p_f^{-1}(g) : p_f^{-1}(R) \to p_f^{-1}(R')$ st

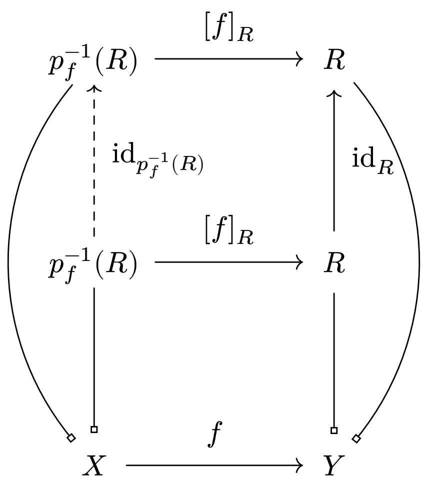
$$p\Big(p_f^{-1}(g)\Big) = \mathrm{id}_X$$

$$[f]_{R'} \circ p_f^{-1}(g) = g \circ [f]_R$$

ie. the following diagram commutes



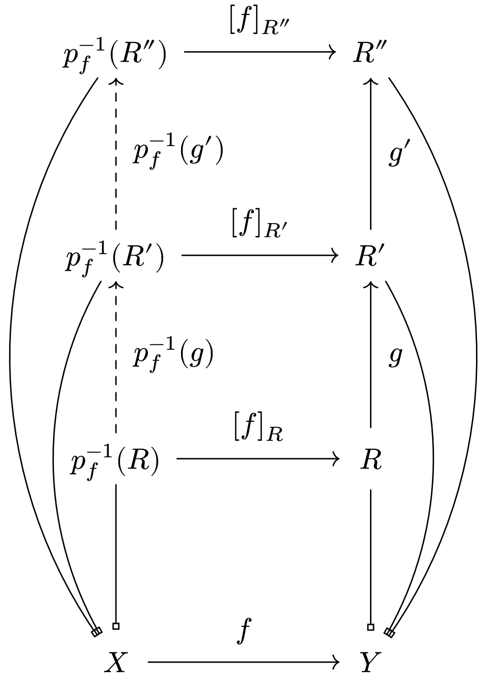
Let us indeed check that this defines a functor. For any $R : p_Y^{-1}$, note that



$\mathrm{id}_{p_f^{-1}(R)}$ satisfies the universal property of $p_f^{-1}(\mathrm{id}_R)$, so we have

$$p_f^{-1}(\mathrm{id}_R) = \mathrm{id}_{p_f^{-1}(R)}$$

Let now $R, R', R'' : p_Y^{-1}$, $g : R \to R'$ and $g' : R' \to R''$.



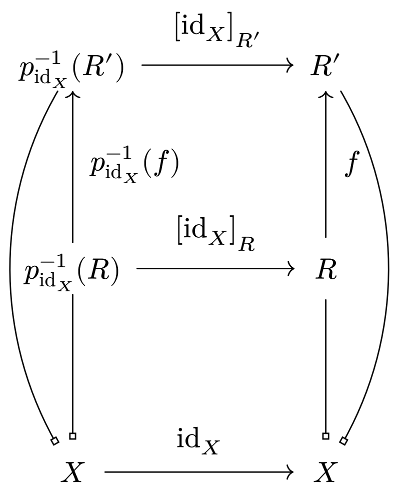
Note that $p_f^{-1}(g') \circ p_f^{-1}(g)$ satisfies the universal property of $p_f^{-1}(g' \circ g)$, so we have

$$p_f^{-1}(g' \circ g) = p_f^{-1}(g') \circ p_f^{-1}(g)$$

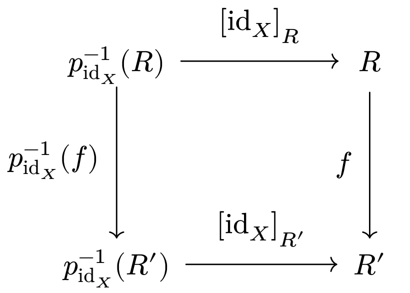
Let us now show that p^{-1} indeed defines a pseudo-functor.

J.1.1.2 Pseudo identity law

For $X : \mathscr{B}^{\mathsf{op}}$, let us first exhibit a natural isomorphism $p_{\mathrm{id}_X}^{-1} \tilde{\Rightarrow} \mathrm{id}_X$. For $R : p_X^{-1}$, we have $[\mathrm{id}_X]_R :$ $p_{\mathrm{id}_X}^{-1}(R) \to \mathrm{id}_X(R)$. This defines a natural transformation. Indeed, for $R, R' : p_X^{-1}$ and $f : R \to R'$, the following diagram commutes by definition of $p_{\mathrm{id}_X}^{-1}(f)$:

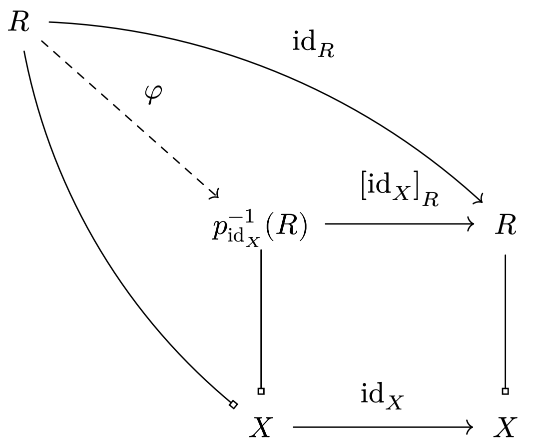


So in particular the upper square commutes



hence $[\mathrm{id}_X]$ is natural. Let's show that each component is an isomorphism.

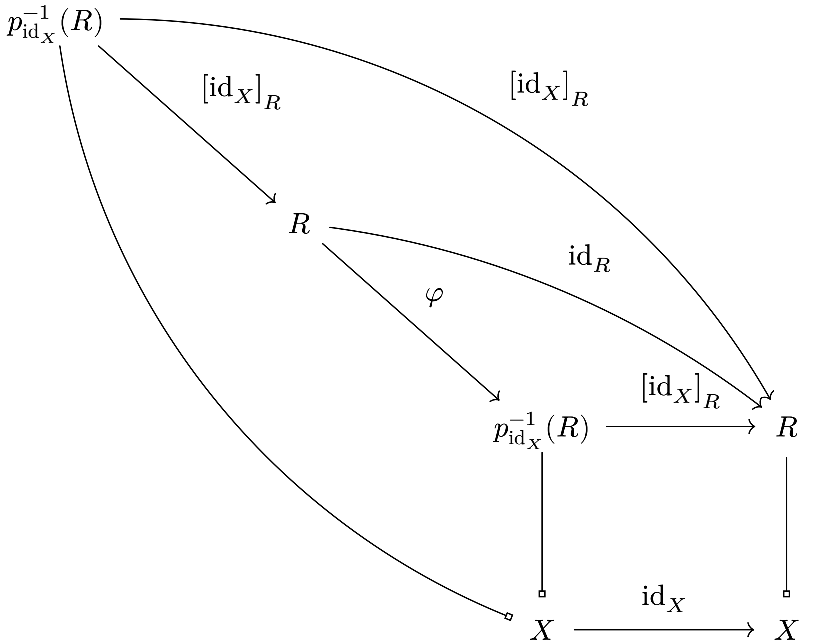
There is a unique morphism $\varphi : R \to p^{-1}_{\mathrm{id}_X}(R)$ making the following diagram commute



So

$$[\mathrm{id}_X]_R \circ \varphi = \mathrm{id}_R$$

Furthermore, the following diagram commutes



Meaning that $\varphi \circ [\mathrm{id}_X]_R$ satisfies the universal property of $[\mathrm{id}_X]_R$ with respect to $[\mathrm{id}_X]_R$. But so does the identity, so, by unicity, we have

$$\varphi \circ [\mathrm{id}_X]_R = \mathrm{id}_{p^{-1}_{\mathrm{id}_X}(R)}$$

Hence $[\mathrm{id}_X]_R$ is an iso.

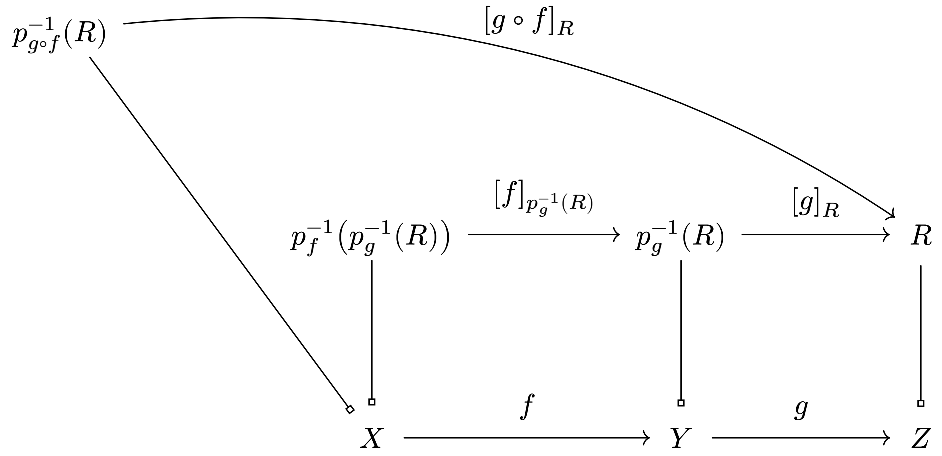
J.1.1.3 Pseudo-composition law

Lemma J.1.1 (Pseudo-composition law)

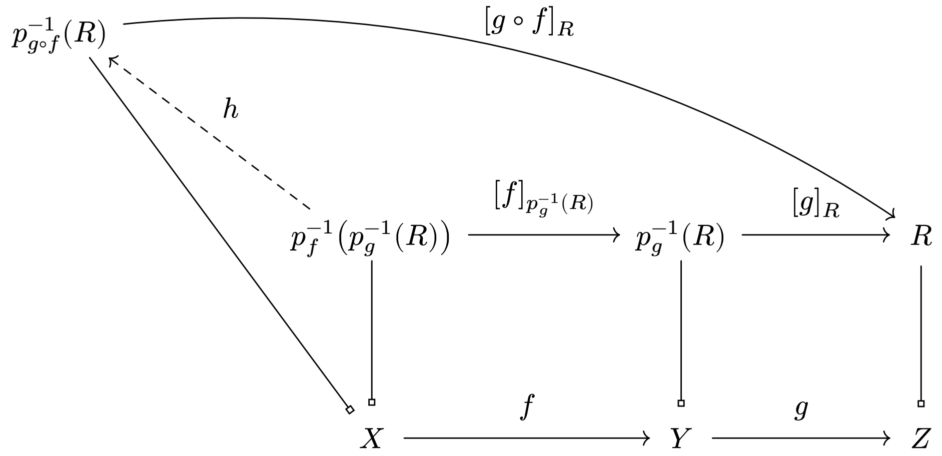
Let $X, Y, Z : \mathcal{B}$, and $f : X \to Y$, $g : Y \to Z$. There is a natural isomorphism

$$[f, g] : p_{g\circ f}^{-1} \Longrightarrow p_f^{-1} \circ p_g^{-1}$$

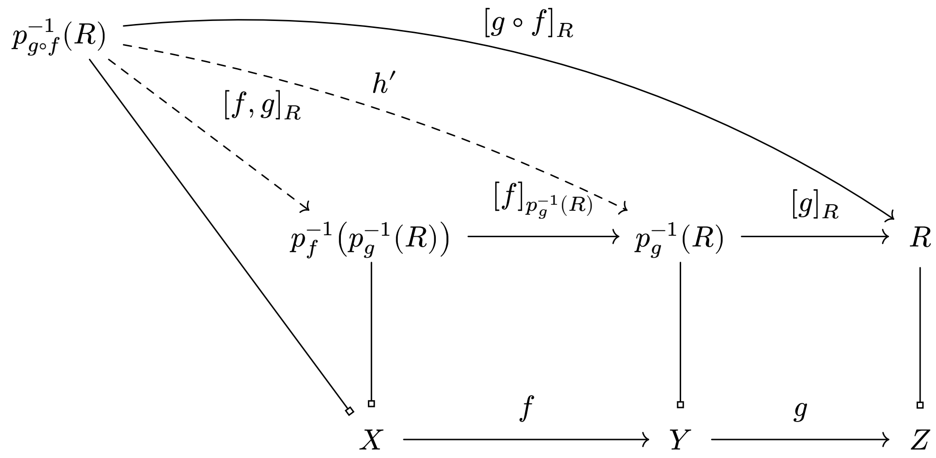
Proof. Let $R : p_Z^{-1}$, and consider the following diagram



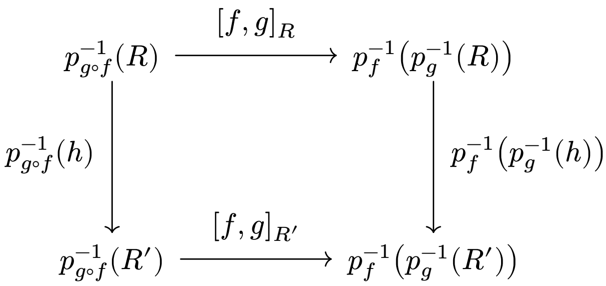
The fact that $[g \circ f]_R$ is cartesian gives a unique morphism $h : p_f^{-1}(p_g^{-1}(R)) \to p_{g\circ f}^{-1}(R)$ making the diagram commute:



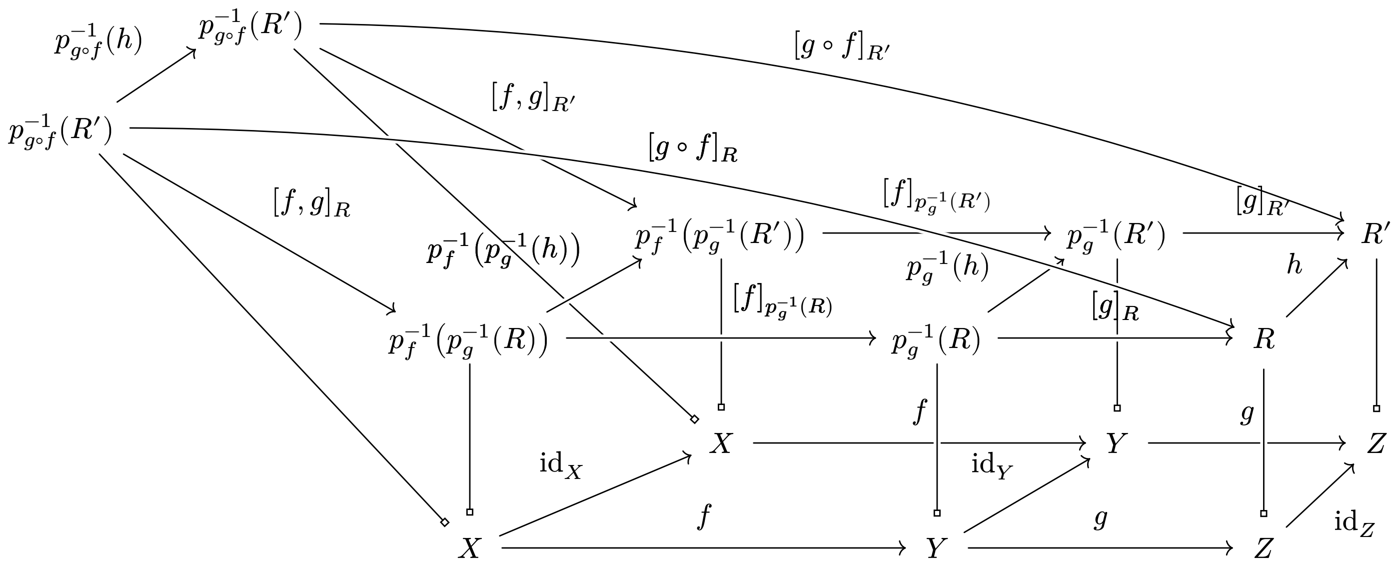
Conversely, the cartesianity of $[g]_R$, and then $[f]_{p_g^{-1}(R)}$ gives $h' : p_{g\circ f}^{-1}(R) \to p_g^{-1}(R)$, then $[f, g]_R : p_{g\circ f}^{-1}(R) \to p_f^{-1}(p_g^{-1}(R))$ making the following commute



In particular, $[f,g]_R$ and h must be each other's inverse. We have to show that this construction is natural. Let $R, R' : p_Z^{-1}$ and $h : R \to R'$. We want to show that the following diagram commutes



Note that in the following diagram



$p_{g\circ f}^{-1}(h)$ is the unique solution to the universal problem of living in the fiber above X and making the top-most square commute. Hence, to prove that

$$[f,g]_{R'} \circ p_{g\circ f}^{-1}(h) = p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R$$

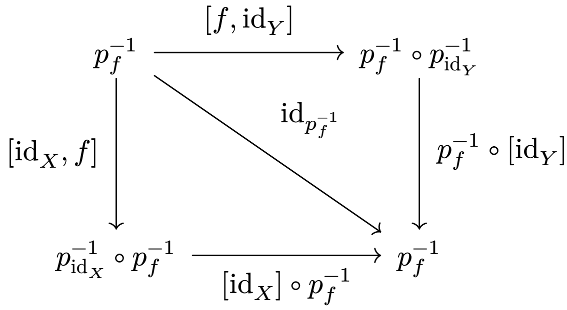
it suffices to show that $[f,g]_{R'}^{-1} \circ p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R$ also satisfies this universal property. Each of these three morphisms lives in the fiber above X, so so does their composition. Furthermore,

$$\begin{aligned}
[g \circ f]_{R'} \circ [f,g]_{R'}^{-1} \circ p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R &= [g]_{R'} \circ [f]_{p_g^{-1}(R')} \circ p_f^{-1}\big(p_g^{-1}(h)\big) \circ [f,g]_R && \text{by definition of } [f,g]_{R'} \\
&= [g]_{R'} \circ p_g^{-1}(h) \circ [f]_{p_g^{-1}(R)} \circ [f,g]_R && \text{by definition of } p_f^{-1}\big(p_g^{-1}(h)\big) \\
&= h \circ [g]_R \circ [f]_{p_g^{-1}(R)} \circ [f,g]_R && \text{by definition of } p_g^{-1}(h) \\
&= h \circ [g \circ f]_R && \text{by definition of } [f,g]_R
\end{aligned}$$

□

J.1.1.4 Identity/composition coherence

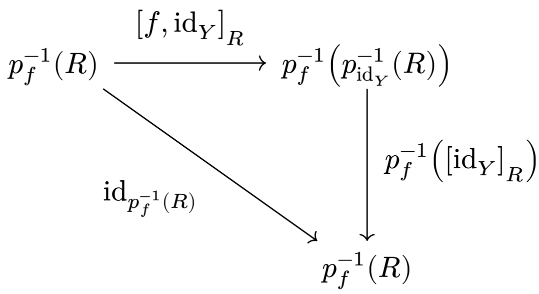
Let $X, Y : \mathscr{B}$ and $f : X \to Y$ We have to check that the following diagram commutes



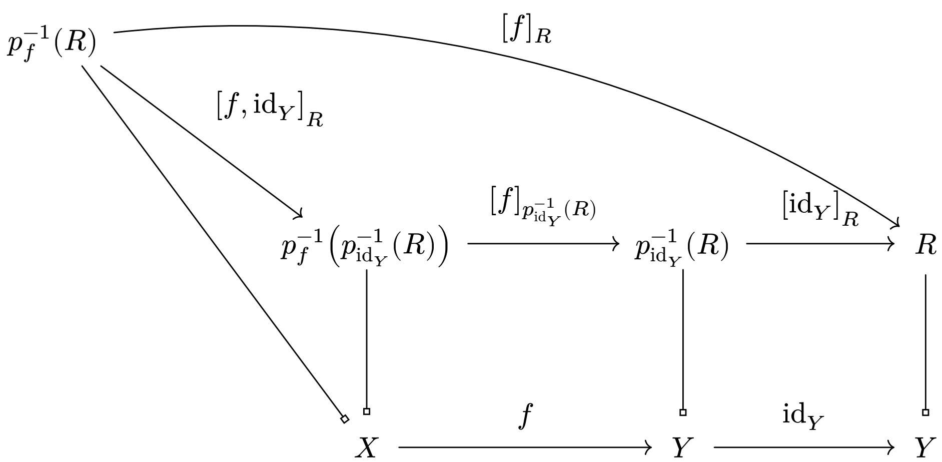
Let's show that each triangle commutes independently.

J.1.1.4.1 Upper triangle

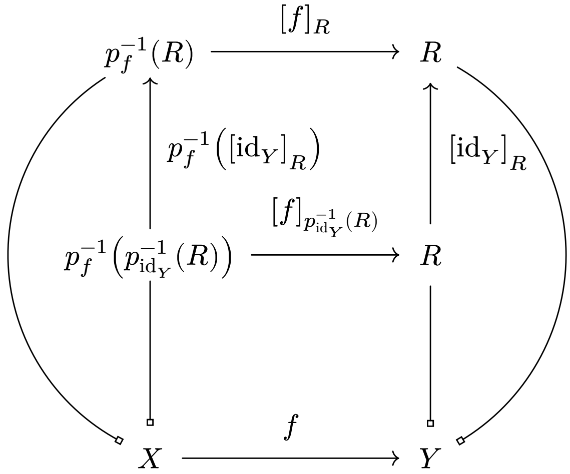
Let $R : p_Y^{-1}$. We have to check the commutation of the following diagram



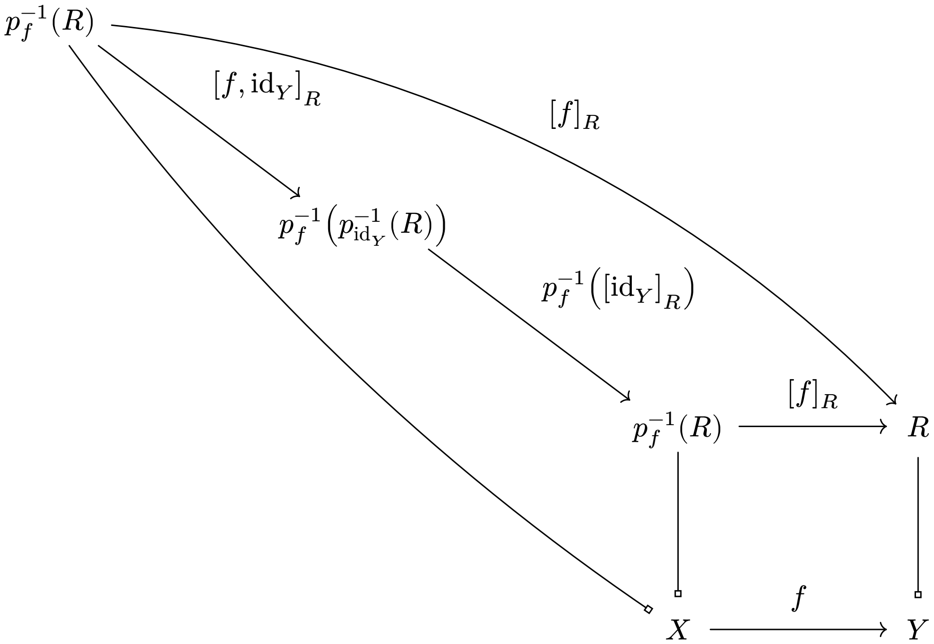
By definition of $[f, \mathrm{id}_Y]$, the following diagram commutes



and we also have



Hence, by stitching the two together, we have that the following diagram commutes

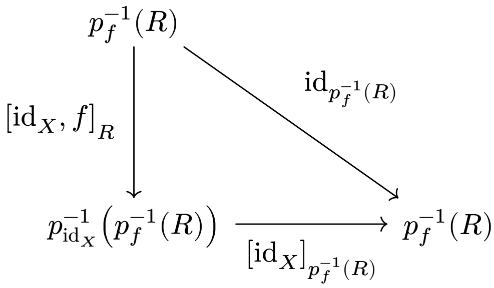


By cartesianity of $[f]_R$, $p_f^{-1}\big([\mathrm{id}_Y]_R\big) \circ [f, \mathrm{id}_Y]_R$ is unique making this diagram commute; but since so does $\mathrm{id}_{p_f^{-1}(R)}$, we must have

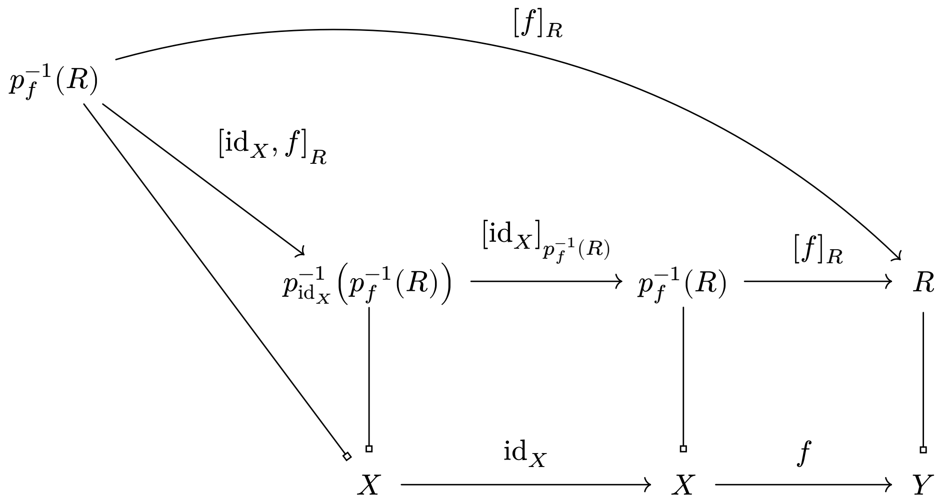
$$p_f^{-1}\big([\mathrm{id}_Y]_R\big) \circ [f, \mathrm{id}_Y]_R = \mathrm{id}_{p_f^{-1}(R)}$$

J.1.1.4.2 Lower triangle

Let $R : p_Y^{-1}$. We have to show the commutation of the following diagram



By definition of $[\mathrm{id}_X, f]$, the following diagram commutes



but, by cartesianity of $[f]_R$, $[\mathrm{id}_X]_{p_f^{-1}(R)} \circ [\mathrm{id}_X, f]_R$ is unique making this diagram commute. Because $\mathrm{id}_{p_f^{-1}(R)}$ also makes it commute, we must have

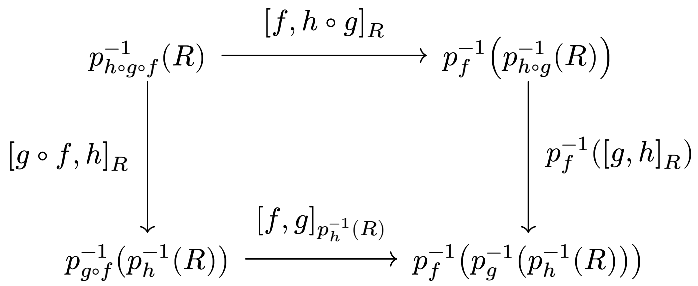
$$[\mathrm{id}_X]_{p_f^{-1}(R)} \circ [\mathrm{id}_X, f]_R = \mathrm{id}_{p_f^{-1}(R)}$$

J.1.1.5 Composition/composition coherence

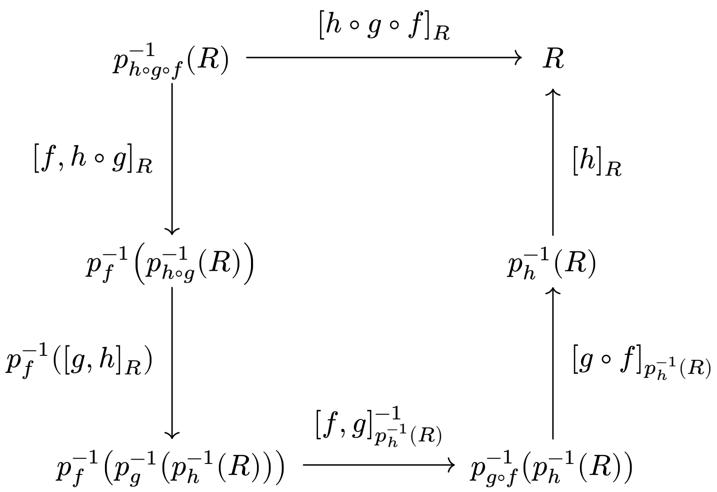
Let $W, X, Y, Z : \mathcal{B}$ and

$$W \xrightarrow{\;f\;} X \xrightarrow{\;g\;} Y \xrightarrow{\;h\;} Z$$

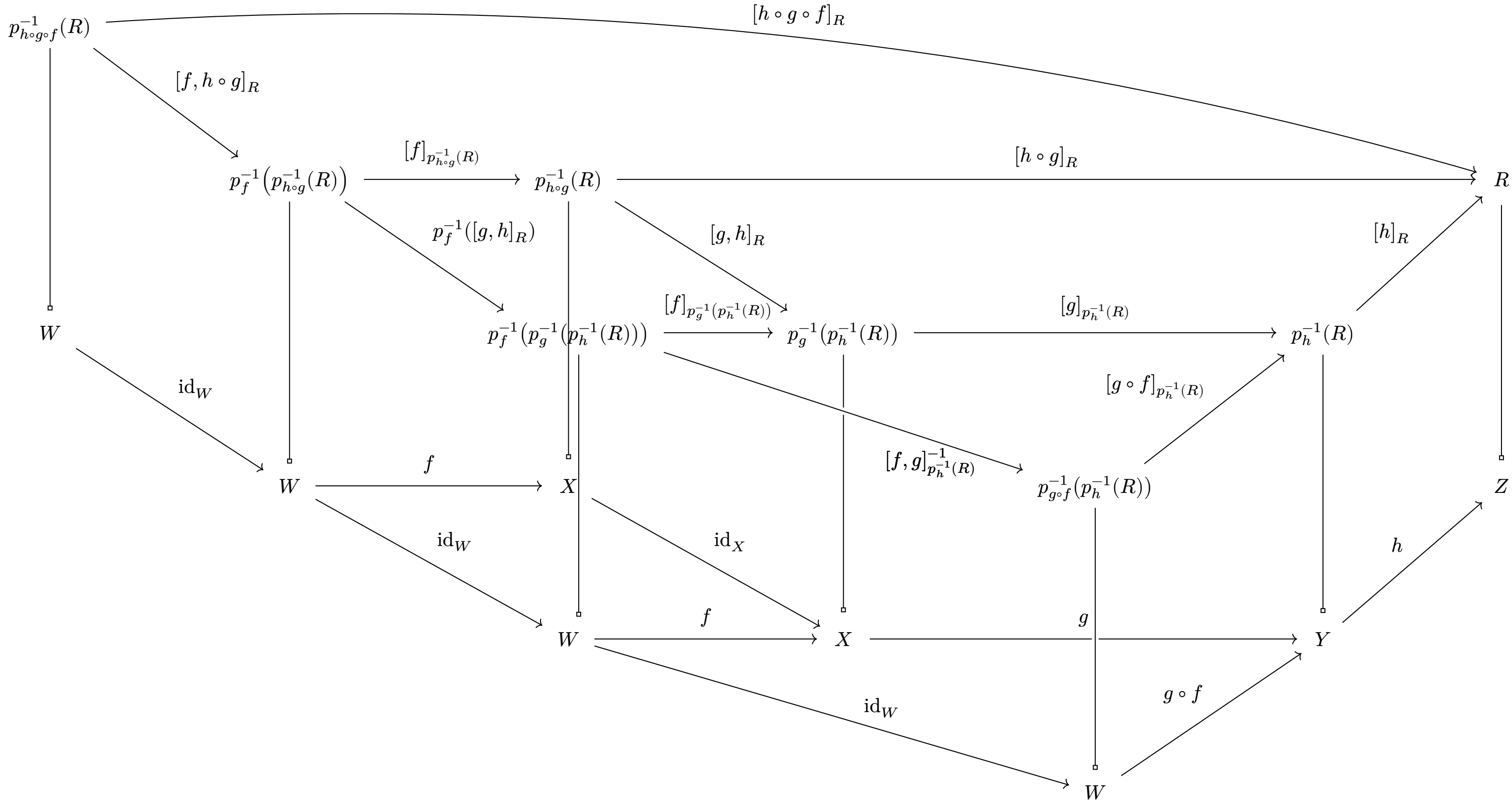
Let $R : p_Z^{-1}$, we want to show that the following diagram commutes



It suffices to show that $[f, g]_{p_h^{-1}(R)}^{-1} \circ p_f^{-1}([g, h]_R) \circ [f, h \circ g]_R$ satisfies the universaly property of $[g \circ f, h]_R$, that is, the following diagram commutes



In the following diagram, each inner diagram commutes, hence the outermost diagram commutes



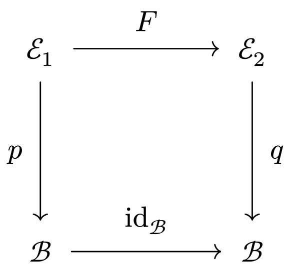
which is exactly what we wanted.

We can therefore define

$$\Phi(p) = p^{-1}$$

J.1.2 Action of Φ on morphisms

Let $p : \mathcal{E}_1 \to \mathcal{B}$ and $q : \mathcal{E}_2 \to \mathcal{B}$ be two fibrations, and $F : p \to q$ be a morphism



We want to define $\nu^F : p^{-1} \to q^{-1}$. Let $X : \mathcal{B}$,

$$\begin{aligned} \nu^F_X : p^{-1}(X) &\longrightarrow q^{-1}(X) \\ S &\longmapsto F(S) \\ f &\longmapsto F(f) \end{aligned}$$

Which is well defined because, if $p(S) = X$, then

$$q(F(S)) = p(S) = X$$

and if $f : S \to R$ is in the fiber above X, then

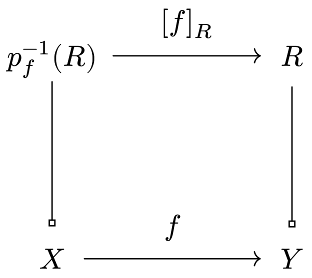
$$q(F(f)) = p(f) = \mathrm{id}_X$$

so $F(f)$ also lives in the fiber above X. ν_X^F is clearly functorial, because F is.

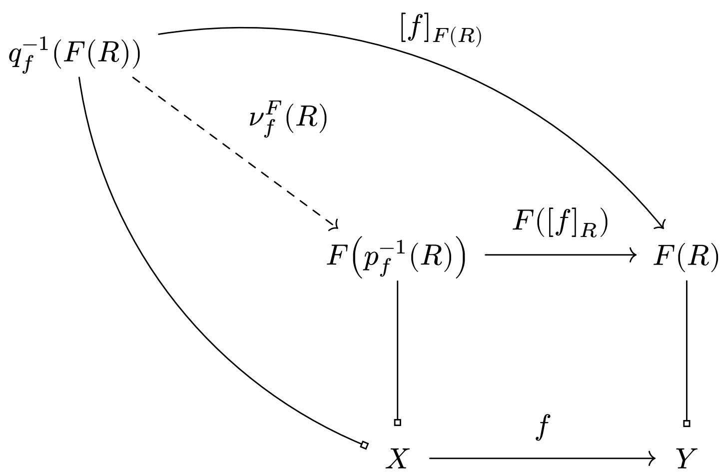
Now, let $f : X \to Y$ in $\mathcal{B}$

$$\nu_f^F(R) : q_f^{-1}(F(R)) \longrightarrow F\left(p_f^{-1}(R)\right)$$

is defined by noting that we have the following commuting diagram



and so, by cartesianity of $F([f]_R)$, which stems from that of $[f]_R$ because F preserves cartesianity,



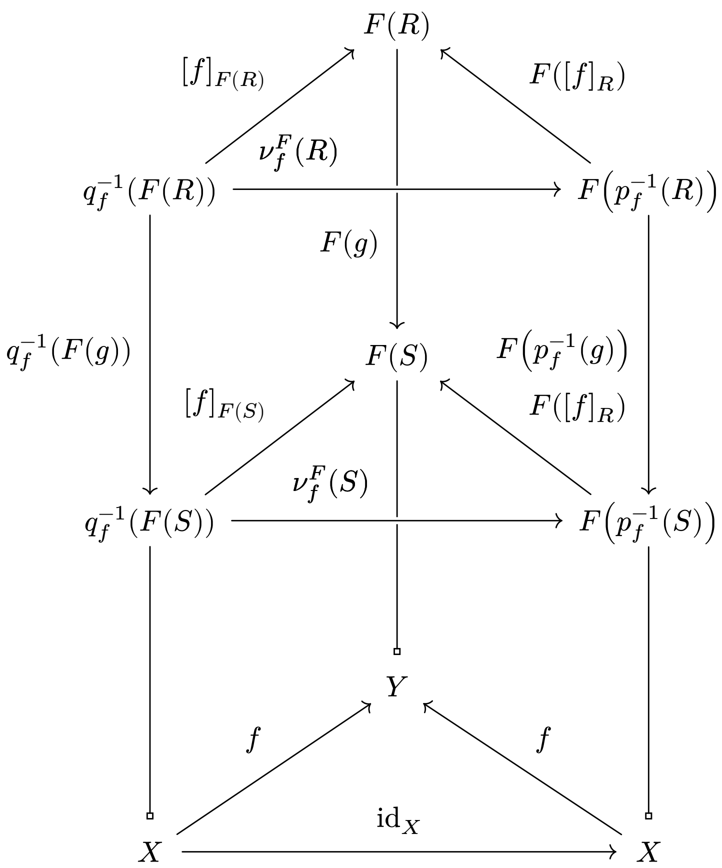
J.1.2.1 ν_f^F is an isomorphism

J.1.2.1.1 Naturality

Lemma J.1.2

ν_f^F is a natural transformation.

Proof. Let $g : R \to S$ be a morphism in p_Y^{-1}.



We have to show that the upper front square commutes. This stems from the fact that $q_f^{-1}(F(g))$ has the universal property of living in the fiber over X, and making the left-most square commute, so we just need to check that the same is true for

$$\nu_f^F(S)^{-1} \circ F\big(p_f^{-1}(g)\big) \circ \nu_f^F(R)$$

which is true because the two triangles and the right-most square commute in the above diagram. □

J.1.2.1.2 Coherences

Lemma J.1.3

ν^F is a morphism.

Proof. We have shown that, for any f, ν_f^F is a natural transformation. We just have to check that ν^F satisfies the coherence conditions.

- Let $X : \mathcal{B}$. Let $R : p_X^{-1}$. We have to check that

$$\mathrm{id}_{\nu_X(R)} = \left(\nu_X^F\left([\mathrm{id}_X]_R\right)\right) \circ \nu_{\mathrm{id}_X}^F(R) \circ [\mathrm{id}_X]_{\nu_X(R)}^{-1}$$

  that is,

$$[\mathrm{id}_X]_{F(R)} = F\left([\mathrm{id}_X]_R\right) \circ \nu_{\mathrm{id}_X}^F(R)$$

  which is, in diagrammatic form,

$$\begin{array}{ccc} q^{-1}_{\mathrm{id}_X}(F(R)) & & \\ {\scriptstyle \nu^F_{\mathrm{id}_X}(R)}\big\downarrow & \searrow^{[\mathrm{id}_X]_{F(R)}} & \\ F\big(p^{-1}_{\mathrm{id}_X}(R)\big) & \xrightarrow[F\left([\mathrm{id}_X]_R\right)]{} & F(R) \end{array}$$

the commutation of this diagram is exactly the definition of $\nu^F_{\mathrm{id}_X}(R)$.

- Let $X, Y, Z : \mathcal{B}$ be three objects, $f : X \to Y$ and $g : Y \to Z$ be two morphisms in $\mathcal{B}$. Let $R : p_Z^{-1}$. We have to check that

$$\nu^F_{g\circ f}(R) = \nu^F_X([f,g]^{-1}_R) \circ \nu^F_f\big(p_g^{-1}(R)\big) \circ q_f^{-1}\big(\nu^F_g(R)\big) \circ [f,g]'_{\nu^F_Z(R)}$$

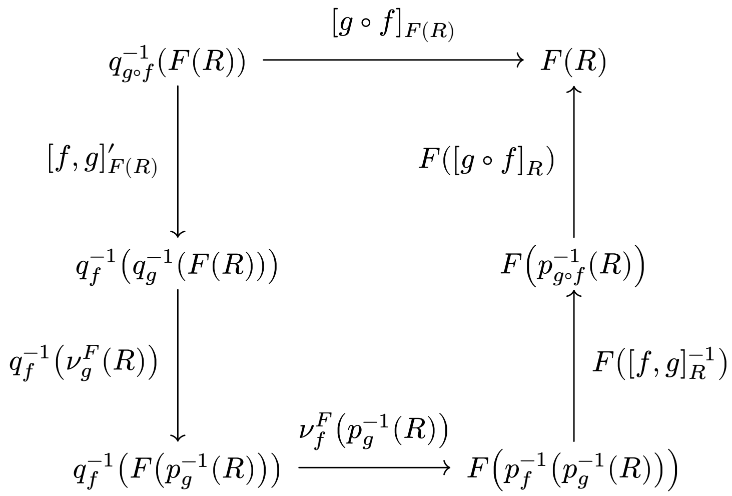
that is, that the following diagram commutes

$$\begin{array}{ccc} q^{-1}_{g\circ f}(F(R)) & \xrightarrow{[f,g]'_{\nu^F_Z(R)}} & q_f^{-1}\big(q_g^{-1}(F(R))\big) \\ & & \big\downarrow {\scriptstyle q_f^{-1}\left(\nu^F_g(R)\right)} \\ {\scriptstyle \nu^F_{g\circ f}(R)}\big\downarrow & & q_f^{-1}\big(F\big(p_g^{-1}(R)\big)\big) \\ & & \big\downarrow {\scriptstyle \nu^F_f\left(p_g^{-1}(R)\right)} \\ F\big(p^{-1}_{g\circ f}(R)\big) & \xrightarrow{F([f,g]_R)} & F\big(p_f^{-1}\big(p_g^{-1}(R)\big)\big) \end{array}$$

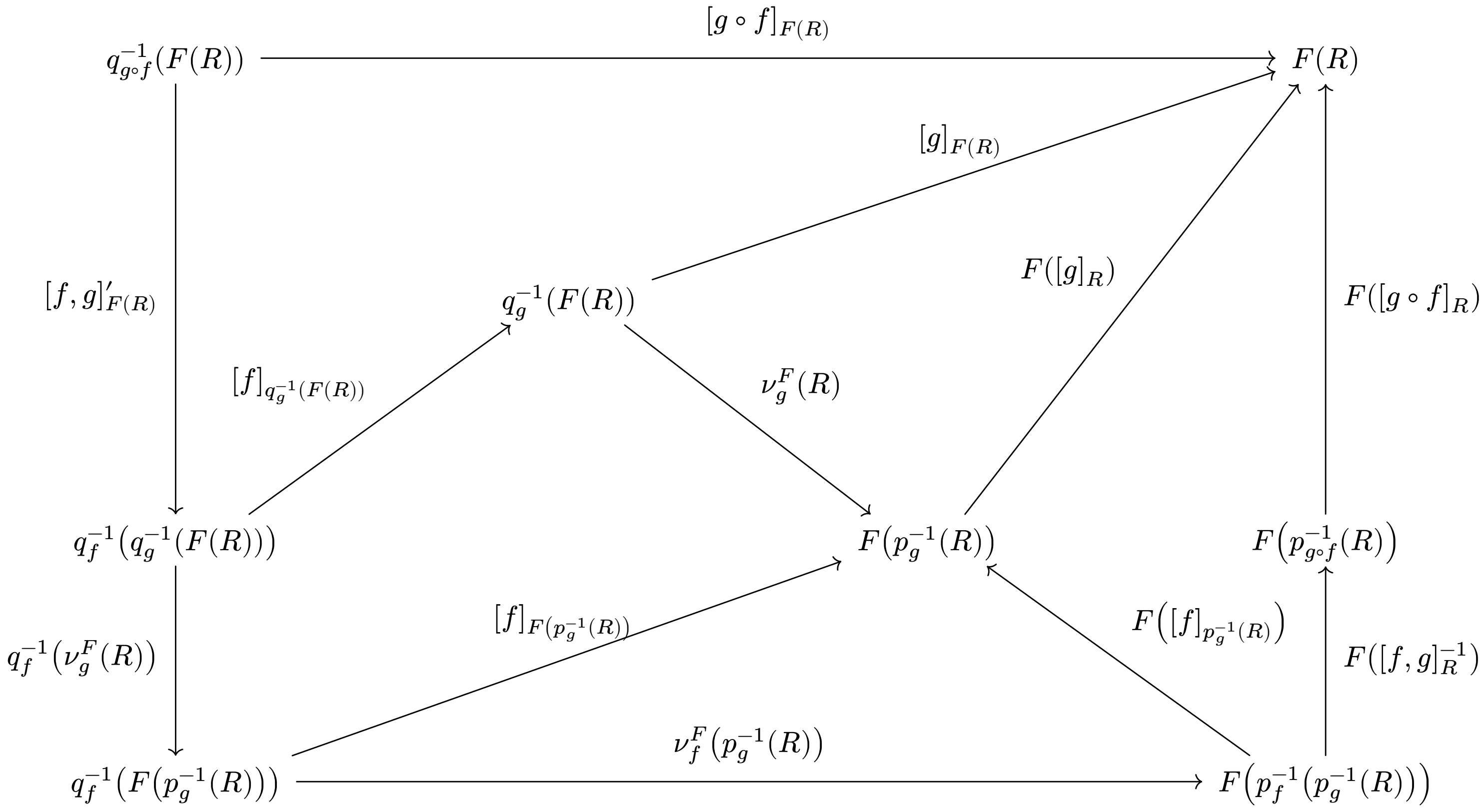
$\nu^F_{g\circ f}(R)$ is defined as the unique map in the fiber above X that makes the following diagram commute

$$\begin{array}{ccc} q^{-1}_{g\circ f}(F(R)) & & \\ {\scriptstyle \nu^F_f(R)}\big\downarrow & \searrow^{[g\circ f]_{F(R)}} & \\ F\big(p^{-1}_{g\circ f}(R)\big) & \xrightarrow[F([g\circ f]_R)]{} & F(R) \end{array}$$

Hence, we just need to show that the following diagram commutes



Indeed, we can fill it with commuting diagrams as follows



□

J.1.2.1.3 Iso

Lemma J.1.4

$\nu_f^F(R)$ is an isomorphism.

Proof. This stems from the fact that $[f]_{F(R)}$ is cartesian. □

We therefore define

$$\Phi(F) = \nu^F$$

J.2 Grothendieck construction

In this section, we will define a functor $\Psi : \mathbf{Pfct}_{\mathcal{B}} \to \mathbf{Fib}_{\mathcal{B}}$.

J.2.1 Action of Ψ on objects

Let $\mathcal{P} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ be a pseudo-functor. Let's build a fibration over B out of it.

Definition J.2.1 (Total category)

The total category $\int \mathcal{P}$ has

- objects: pairs (X, x) with $X : \mathcal{B}^{\mathsf{op}}$ and $x : \mathcal{P}_X$;
- morphisms between two objects (A, a) and (B, b): pairs (f_1, f_2) with $f_1 : A \to B$ in $\mathcal{B}$ and $f_2 : a \to \mathcal{P}_{f_1}(b)$.
- identities for $(X, x) : \int \mathcal{P}$: $\left(\mathrm{id}_{(X,x)}\right)_0 = \mathrm{id}_X$ and

$$\left(\mathrm{id}_{(X,x)}\right)_1 : x \longrightarrow \mathcal{P}_{\mathrm{id}_X}(x)$$
$$\left(\mathrm{id}_{(X,x)}\right)_1 = i_X^{-1}(x)$$

- composition, given $(A, a), (B, b), (C, c) : \int \mathcal{P}$, $(f_1, f_2) : (A, a) \to (B, b)$ and $(g_1, g_2) : (B, b) \to (C, c)$: $(h_1, h_2) = (g_1, g_2) \circ (f_1, f_2)$ by
  - $h_1 : A \to C = g_1 \circ f_1$
  - $h_2 : a \to \mathcal{P}_{g_1 \circ f_1}(c)$ by

$$a \xrightarrow{f_2} \mathcal{P}_{f_1}(b) \xrightarrow{\mathcal{P}_{f_1}(g_2)} \mathcal{P}_{f_1}\left(\mathcal{P}_{g_1}(c)\right) \xrightarrow{c^{-1}_{f_1,g_1}(c)} \mathcal{P}_{g_1 \circ f_1}(c)$$

♣

Lemma J.2.2

Let f be an isomorphism in $\int \mathcal{P}$. f_1 and f_2 are invertible.

♡

Proof. Let $(f_1, f_2) : (A, a) \to (B, b)$ a morphism in $\int \mathcal{P}$, and $(g_1, g_2) : (B, b) \to (A, a)$ such that

$$(f_1, f_2) \circ (g_1, g_2) = \mathrm{id}_{B,b}$$
$$(g_1, g_2) \circ (f_1, f_2) = \mathrm{id}_{A,a}$$

We have that $f_1 \circ g_1 = \mathrm{id}_B$ and $g_1 \circ f_1 = \mathrm{id}_A$, so f_1 is invertible and $f_1^{-1} = g_1$.

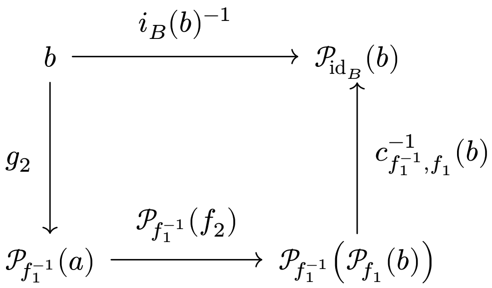
Furthermore, the following diagram commute

$$\begin{array}{ccc} a & \xrightarrow{i_A(a)^{-1}} & \mathcal{P}_{\mathrm{id}_A}(a) \\ {\scriptstyle f_2}\downarrow & & \uparrow{\scriptstyle c_{f_1,f_1^{-1}}(a)^{-1}} \\ \mathcal{P}_{f_1}(b) & \xrightarrow{\mathcal{P}_{f_1}(g_2)} & \mathcal{P}_{f_1}\left(\mathcal{P}_{f_1^{-1}}(a)\right) \end{array}$$

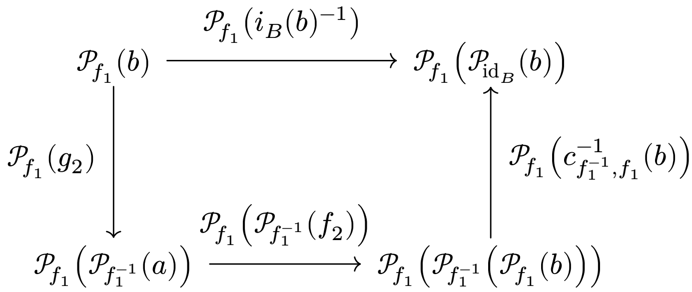
So we have a candidate for the inverse of f_2, namely,

$$\hat{f}_2 := i_A(a) \circ c^{-1}_{f_1,f_1^{-1}}(a) \circ \mathcal{P}_{f_1}(g_2)$$

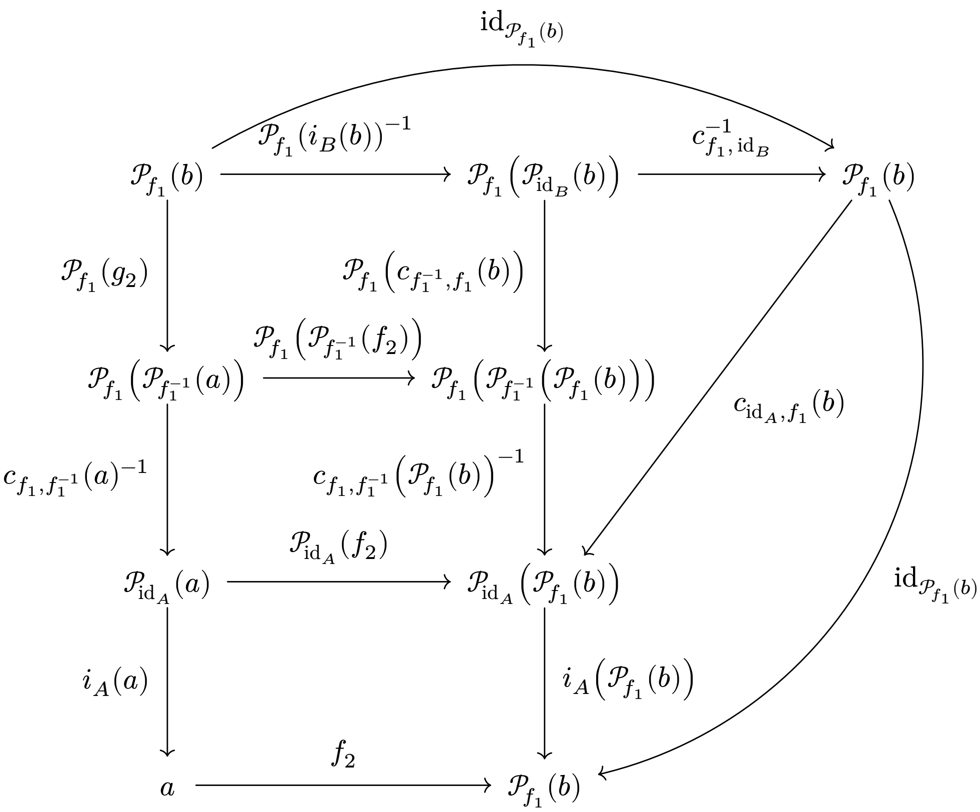
because the diagram above states that $\hat{f}_2 \circ f_2 = \mathrm{id}_a$. Furthermore, the following diagram commutes



Hence so does its image by $\mathcal{P}_{f_1}$



Thus the following diagram commutes (the other inner squares/triangles are coherence conditions)



the outermost diagram states precisely

$$f_2 \circ \hat{f}_2 = \mathrm{id}_{\mathcal{P}_{f_1}(b)}$$

□

Definition J.2.3 (Forgetful fibration)

We can now define the forgetful fibration

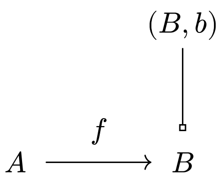
$$\begin{aligned}\pi(\mathcal{P}) : \quad \int \mathcal{P} &\longrightarrow \mathcal{B} \\ (A, a) &\longmapsto A \\ (f_1, f_2) &\longmapsto f_1\end{aligned}$$

which is clearly functorial. ♣

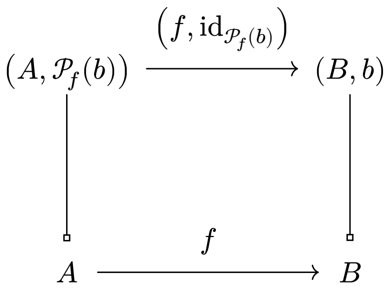
Lemma J.2.4

The forgetful fibration is a fibration. ♡

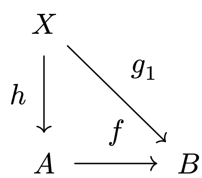
Proof. Let $A, B : \mathcal{B}$, $f : A \to B$ and $b : \mathcal{P}_B$, ie we have the following diagram:



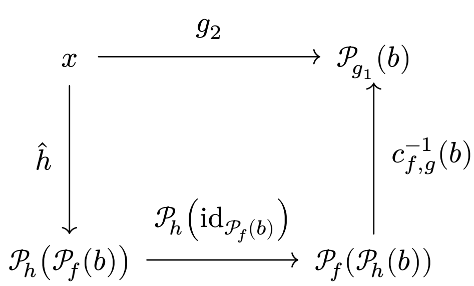
We can lift f as



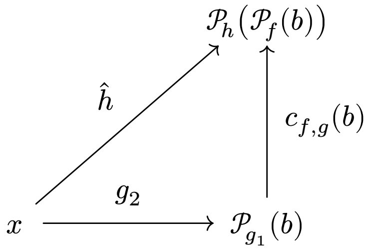
Hence, we just have to show that $\left(f, \mathrm{id}_{\mathcal{P}_f(b)}\right)$ is cartesian. Let $X : \mathcal{B}$, $x : X$, $(g_1, g_2) : (X, x) \to (B, b)$ and $h : X \to A$ such that $g_1 = f \circ h$:



We have to show there is a unique $\hat{h} : x \to \mathcal{P}_h\left(\mathcal{P}_f(b)\right)$ with



which is equivalent to the following diagram commuting



but it is obvious that there is exactly one $\hat{h}$ that makes this commute, namely,

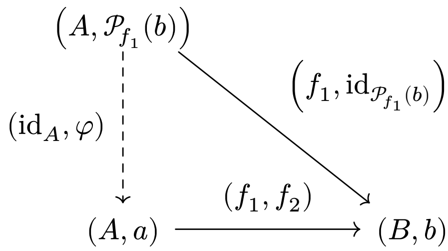
$$\hat{h} = c_{f,g}(b) \circ g_2$$

$\square$

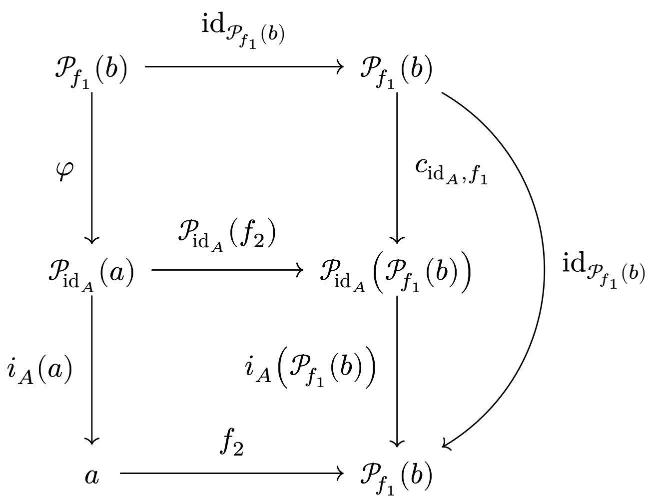
Lemma J.2.5

Let (f_1, f_2) be a cartesian morphism in $\int \mathcal{P}$. f_2 is an isomorphism.

Proof. Let $(f_1, f_2) : (A, a) \to (B, b)$ be a cartesian morphism. In the previous proof, we have established that $\left(f_1, \mathrm{id}_{\mathcal{P}_{f_1}(b)}\right)$ is cartesian. Hence, there exists a unique isomorphism (id_A, φ) making the following diagram commute



hence the top square of this diagram commutes



the lower square commutes by naturality of i_A, and the triangle commutes by a coherence condition. Therefore,

$$f_2 \circ i_A(a) \circ \varphi = \mathrm{id}_{\mathcal{P}_{f_1}(b)}$$

By Lemma J.2.2, (id_A, φ) is an isomorphism, so φ is too and hence

$$f_2 = (i_A(a) \circ \varphi)^{-1}$$

so f_2 is an isomorphism. □

Lemma J.2.6

Let (f_1, f_2) be a morphism in $\int \mathcal{P}$, with f_2 an isomorphism. (f_1, f_2) is cartesian. ♡

Proof. Let $(f_1, f_2) : (X, x) \to (Y, y)$, with $f_2 : x \to \mathcal{P}_{f_1}(y)$ an isomorphism, $(g_1, g_2) : (Z, z) \to (Y, y)$ and $h : Z \to X$ such that $g_1 = f_1 \circ h$. We want to find a unique $\hat{h} : z \to \mathcal{P}_h(x)$ such that $(g_1, g_2) = (f_1, f_2) \circ \left(h, \hat{h}\right)$, which is equivalent to the commutation of the following diagram

$$\begin{array}{ccc} z & \xrightarrow{\quad g_2 \quad} & \mathcal{P}_{g_1}(y) \\ {\scriptstyle \hat{h}}\,\Big\downarrow & & \Big\downarrow\,{\scriptstyle c_{h,f_1}(y)} \\ \mathcal{P}_h(x) & \xrightarrow{\quad \mathcal{P}_h(f_2) \quad} & \mathcal{P}_h\left(\mathcal{P}_{f_1}(y)\right) \end{array}$$

Since f_2 is an iso, it is clear that there is a unique $\hat{h}$ making the above diagram commute. □

We thus define Ψ on objects by

$$\Psi(\mathcal{P}) = \left(\int \mathcal{P}, \pi(\mathcal{P})\right)$$

J.2.2 Action of Ψ on morphisms

Let $\mathcal{P}$, $\mathcal{P}'$ be two pseudo-functors, and $\nu : \mathcal{P} \to \mathcal{P}'$ a morphism in $\mathbf{Pfct}_{\mathcal{B}}$.

Definition J.2.7

Let

$$\begin{aligned} F_\nu : \quad & \int \mathcal{P} \longrightarrow \int \mathcal{P}' \\ & (X, x) \longmapsto (X, \nu_X(x)) \\ & (f_1, f_2) \longmapsto \left(f_1, \nu_{f_1}(b)^{-1} \circ \nu_X(f_2)\right) \end{aligned}$$

That is, for $(f_1, f_2) : (A, a) \to (B, b)$, we have

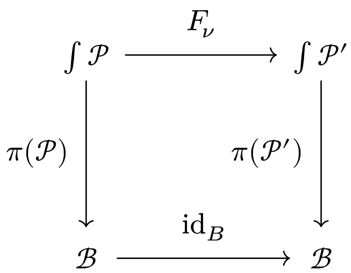
$$\begin{array}{ccc} \nu_X(a) & \xrightarrow{\quad \nu_A(f_2) \quad} & \nu_A\left(\mathcal{P}_{f_1}(b)\right) \\ & {\scriptstyle (F_\nu(f_1, f_2))_2}\searrow & \Big\downarrow\,{\scriptstyle \nu_{f_1}(b)^{-1}} \\ & & \mathcal{P}'_{f_1}(\nu_B(b)) \end{array}$$

♣

Lemma J.2.8

F_ν is a fibration morphism ♡

Proof. We have to show that it makes the following diagram commute



and that F_ν preserves the cartisian morphisms.

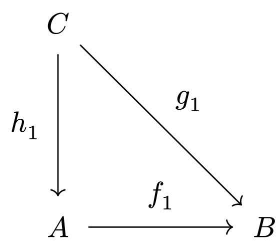
- Let's show the two functors agree:
  - on objects: let $(X, x) : \int \mathcal{P}$,

$$\begin{aligned} \pi(\mathcal{P}')(F_\nu(X, x)) &= \pi(\mathcal{P}')(X, \nu_X(x)) \\ &= X \\ &= \pi(\mathcal{P})(X, x) \end{aligned}$$

  - on morphisms: let $(f_1, f_2) : (A, a) \to (B, b)$,

$$\begin{aligned} \pi(\mathcal{P}')(F_\nu(f_1, f_2)) &= \pi(\mathcal{P}')\left(\left(f_1, \nu_{f_1}(b)^{-1} \circ \nu_A(f_2)\right)\right) \\ &= f_1 \\ &= \pi(\mathcal{P})(f_1, f_2) \end{aligned}$$

  Hence the diagram commutes.
- Let $(f_1, f_2) : (A, a) \to (B, b)$ be a cartesian morphism in $\int \mathcal{P}$. Let $(g_1, g_2) : (C, c) \to (B, \nu_B(b))$ be a morphism in $\int \mathcal{P}'$ and $h_1 : C \to B$ in $\mathcal{B}$ such that the following diagram commutes



  Let's show that there exists a unique $h_2 : c \to \mathcal{P}'_{h_1}(a)$ such that

$$
\begin{array}{ccc}
(C,c) & & \\
\Big\downarrow{\scriptstyle (h_1,h_2)} & \searrow^{(g_1,g_2)} & \\
(A,\nu_A(a)) & \xrightarrow[\left(f_1,\,\nu_{f_1}(b)^{-1}\circ\nu_A(f_2)\right)]{} & (B,\nu_B(b))
\end{array}
$$

that is

$$
\begin{array}{ccc}
c & \xrightarrow{g_2} & \mathcal{P}'_{g_1}(\nu_B(b)) \\
 & & \Big\downarrow{\scriptstyle c'_{h_1,f_1}(\nu_B(b))} \\
\Big\downarrow{\scriptstyle h_2} & & \mathcal{P}'_{h_1}\left(\mathcal{P}'_{f_1}(\nu_B(b))\right) \\
 & & \Big\downarrow{\scriptstyle \mathcal{P}'_{h_1}\left(\nu_{f_1}(b)\right)} \\
\mathcal{P}'_{h_1}(\nu_A(a)) & \xrightarrow{\mathcal{P}'_{h_1}(\nu_A(f_2))} & \mathcal{P}'_{h_1}\left(\nu_A\left(\mathcal{P}_{f_1}(b)\right)\right)
\end{array}
$$

By Lemma J.2.5, f_2 is an isomorphism, hence the commutation of the latter diagram is equivalent to that of the following, for which there clearly exists a unique h_2

$$
\begin{array}{ccc}
c & \xrightarrow{g_2} & \mathcal{P}'_{g_1}(\nu_B(b)) \\
 & & \Big\downarrow{\scriptstyle c'_{h_1,f_1}(\nu_B(b))} \\
\Big\downarrow{\scriptstyle h_2} & & \mathcal{P}'_{h_1}\left(\mathcal{P}'_{f_1}(\nu_B(b))\right) \\
 & & \Big\downarrow{\scriptstyle \mathcal{P}'_{h_1}\left(\nu_{f_1}(b)\right)} \\
\mathcal{P}'_{h_1}(\nu_A(a)) & \xleftarrow{\mathcal{P}'_{h_1}(\nu_A(f_2^{-1}))} & \mathcal{P}'_{h_1}\left(\nu_A\left(\mathcal{P}_{f_1}(b)\right)\right)
\end{array}
$$

□

J.3 The equivalence

J.3.1 $\Phi \circ \Psi$

Let $\mathcal{P} : \mathcal{B}^{\mathsf{op}} \to \mathbf{Cat}$ be a pseudo-functor.

Definition J.3.1

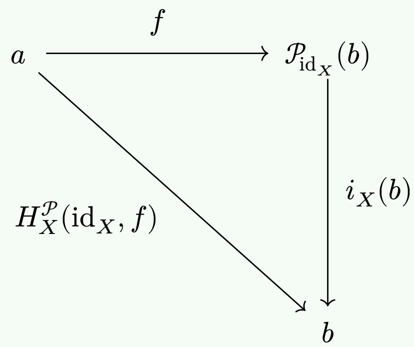
Consider

$$H^{\mathcal{P}} : \pi(\mathcal{P})^{-1} \longrightarrow \mathcal{P}$$

defined by, for $X : \mathcal{B}$,

$$\begin{aligned} H^{\mathcal{P}}_X :\ \pi(\mathcal{P})^{-1}_X &\longrightarrow \mathcal{P}_X \\ (X, x) &\longmapsto x \\ (\mathrm{id}_X, f) &\longmapsto i_X(b) \circ f \end{aligned}$$

Let $(\mathrm{id}_X, f) : (X, a) \to (X, b)$ in $\pi(\mathcal{P})^{-1}_X$, that is, $f : a \to \mathcal{P}_{\mathrm{id}_X}(b)$. We have



♣

Lemma J.3.2

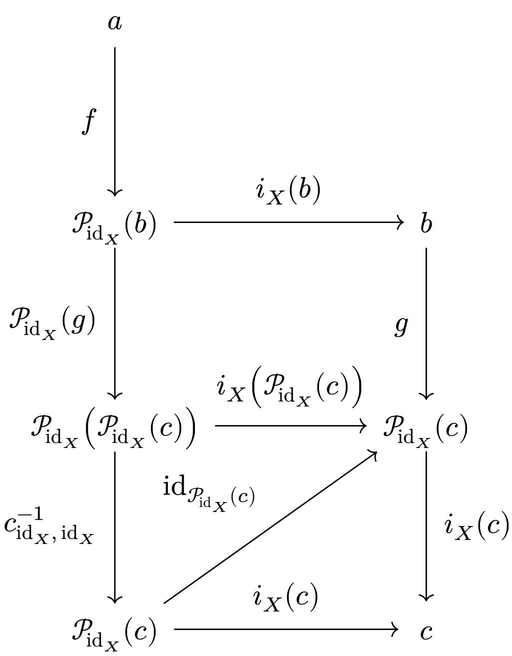
For $X : \mathcal{B}$, $H^{\mathcal{P}}_X$ is a functor. ♡

Proof. Let $(X, x) : \pi(\mathcal{P})^{-1}_X$. $\mathrm{id}_{X,x} = (\mathrm{id}_X, i^{-1}_X(x))$, and so

$$\begin{aligned} H^{\mathcal{P}}_X(\mathrm{id}_{X,x}) &= i_X(x) \circ i^{-1}_X(x) \\ &= \mathrm{id}_x \end{aligned}$$

Furthermore, for $(X, a), (X, b), (X, c) : \pi(\mathcal{P})^{-1}_X$ and $(\mathrm{id}_X, f) : (X, a) \to (X, b)$ and $(\mathrm{id}_X, f) : (X, b) \to (X, c)$,

$$\begin{aligned} H^{\mathcal{P}}_X((\mathrm{id}_X, g) \circ (\mathrm{id}_X, f)) &= H^{\mathcal{P}}_X\left(\mathrm{id}_X, c^{-1}_{\mathrm{id}_X, \mathrm{id}_X}(c) \circ \mathcal{P}_{\mathrm{id}_X}(g) \circ f\right) \\ &= i_X(c) \circ c^{-1}_{\mathrm{id}_X, \mathrm{id}_X}(c) \circ \mathcal{P}_{\mathrm{id}_X}(g) \circ f \end{aligned}$$



the lower right triangle commutes trivially, the triangle above commutes by a composition/identity coherence, and the square above by naturality of i_X. The outer diagram shows that

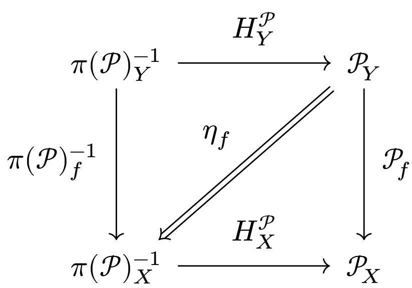
$$i_X(c) \circ c^{-1}_{\mathrm{id}_X,\mathrm{id}_X}(c) \circ \mathcal{P}_{\mathrm{id}_X}(g) \circ f = \underbrace{(i_X(c) \circ g)}_{=H^{\mathcal{P}}_X(\mathrm{id}_X, g)} \circ \underbrace{(i_X(b) \circ f)}_{=H^{\mathcal{P}}_X(\mathrm{id}_X, f)}$$

□

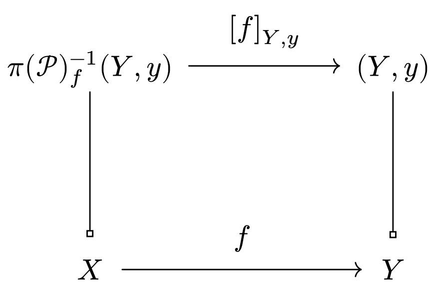
Lemma J.3.3

$H^{\mathcal{P}}$ is a morphism in $\mathbf{Pfct}_{\mathcal{B}}$. ♡

Proof. Let $f : X \to Y$ in $\mathcal{B}$, let's show that there is a natural isomorphism



Let $(Y, y) : \pi(\mathcal{P})^{-1}_Y$, that is, $y : \mathcal{P}_Y$. We have



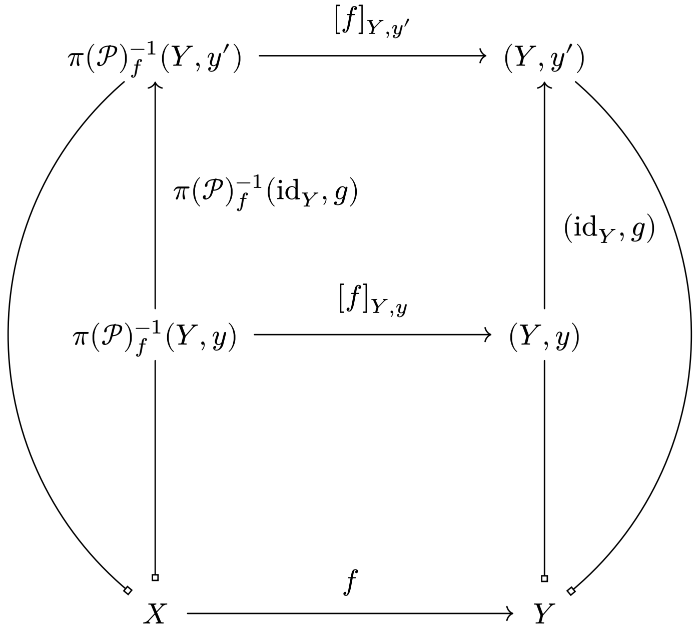
We can write $[f]_{Y,y} = \big(f, \eta_f(y)\big)$ with $\eta_f(y) : H^{\mathcal{P}}_X\Big(\pi(\mathcal{P})^{-1}_f(Y, y)\Big) \to \mathcal{P}_f(y)$.

Let's prove that η_f is natural, and that it is an isomorphism.

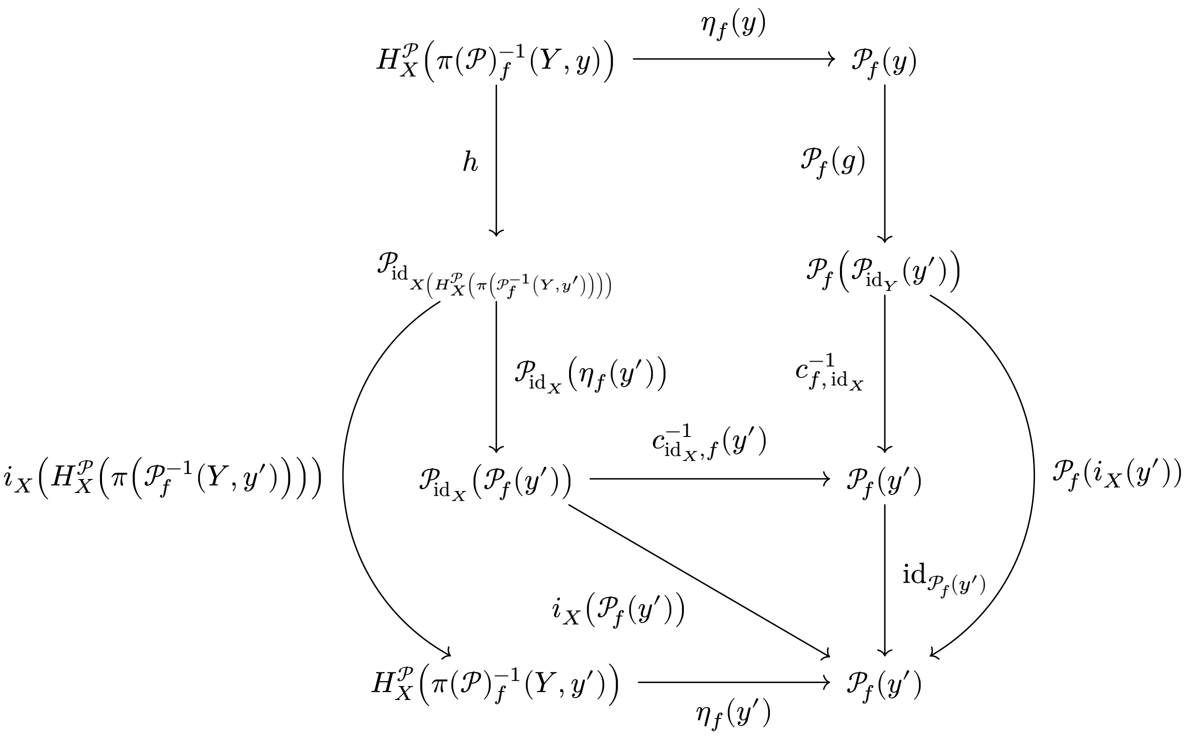
- let $y, y' : \mathcal{P}_Y$ and $g : y \to y'$. We want to show the the following diagram commutes

$$\begin{array}{ccc} H_X^{\mathcal{P}}\big(\pi(\mathcal{P})_f^{-1}(Y, y)\big) & \xrightarrow{\eta_f(y)} & \mathcal{P}_f(y) \\ {\scriptstyle H_X^{\mathcal{P}}\big(\pi(\mathcal{P})_f^{-1}(\mathrm{id}_Y, g)\big)}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_f\big(H_Y^{\mathcal{P}}(\mathrm{id}_Y, g)\big)} \\ H_X^{\mathcal{P}}\big(\pi(\mathcal{P})_f^{-1}(Y, y')\big) & \xrightarrow{\eta_f(y')} & \mathcal{P}_f(y') \end{array}$$

  We have that the following diagram commutes



  We can write $\pi(\mathcal{P})_f^{-1}(\mathrm{id}_Y, g) = (\mathrm{id}_Y, h)$, and so the commutation of the square implies the following commutation on the second component of the morphisms



the two triangles commute by a composition/identity coherence, while the left square is the naturality of i_X. Note that the outermost diagram is exactly the one we were looking for, showing that η_f is natural.

- $\big(f, \eta_f(y)\big) = [f]_{Y,y}$ is cartesian (by definition of $[-]_{\_}$), hence, by Lemma J.2.5, $\eta_f(y)$ is an isomorphism, showing that η_f is a natural isomorphism.

□

Lemma J.3.4

$H^{\mathcal{P}}$ is an isomorphism. ♡

Proof. To show that $H^{\mathcal{P}}$ is a pseudo-natural isomorphism, it is enough to show that each of its components is an isomorphism. Let $X : \mathcal{B}$. It is clear that both actions on objects and on morphisms of $H_X^{\mathcal{P}}$ are invertible. □

Lemma J.3.5

$H^{\mathcal{P}}$ is natural in $\mathcal{P}$. ♡

Proof. Let $\mathcal{P}, \mathcal{P}' : \mathcal{B}^{\mathrm{op}} \to \mathbf{Cat}$ be two pseudo-functors, and $\nu : \mathcal{P} \to \mathcal{P}'$ be a pseudo-natural transformation. We have to show that

$$\begin{array}{ccc} \pi(\mathcal{P})^{-1} & \xrightarrow{H^{\mathcal{P}}} & \mathcal{P} \\ \Big\downarrow{\scriptstyle \nu^{F_\nu}} & & \Big\downarrow{\scriptstyle \nu} \\ \pi(\mathcal{P}'^{-1}) & \xrightarrow{H^{\mathcal{P}'}} & \mathcal{P}' \end{array}$$

Hence, we have to show that the diagram commutes at each point $X : \mathcal{B}$

$$\begin{array}{ccc} \pi(\mathcal{P})^{-1}_X & \xrightarrow{H^{\mathcal{P}}_X} & \mathcal{P}_X \\ \Big\downarrow \nu^{F_\nu}_X & & \Big\downarrow \nu_X \\ \pi(\mathcal{P}'^{-1}_X) & \xrightarrow{H^{\mathcal{P}'}_X} & \mathcal{P}'_X \end{array}$$

Let's check that the functor agree on each object and morphisms:

- let $x : \mathcal{P}_X$.

$$\begin{aligned} H^{\mathcal{P}'_X\left(\nu^{F_\nu}_X(X,x)\right)} &= H^{\mathcal{P}'}_X(F_\nu(X,x)) \\ &= H^{\mathcal{P}'}_X(X, \nu_X(x)) \\ &= \nu_X(x) \\ &= \nu_X\left(H^{\mathcal{P}}_X(X,x)\right) \end{aligned}$$

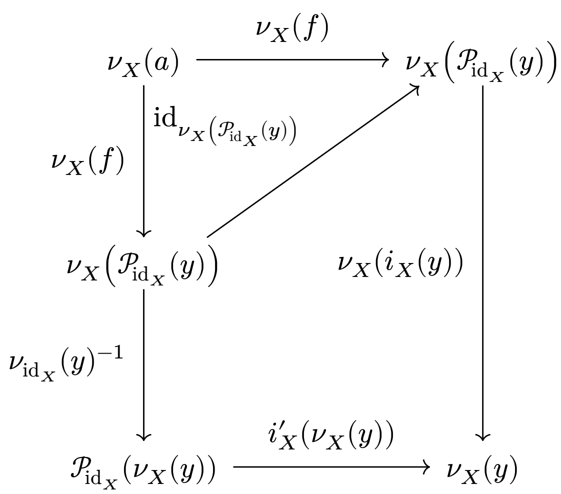
- let $x, y : \mathcal{P}_X$, and $f : x \to \mathcal{P}_{\mathrm{id}_X}(y)$.

$$\begin{aligned} \nu_X\left(H^{\mathcal{P}}_X(\mathrm{id}_X, f)\right) &= \nu_X(i_X(y) \circ f) \\ &= \nu_X(i_X(y)) \circ \nu_X(f) \end{aligned}$$

  and

$$\begin{aligned} H^{\mathcal{P}'}_X\left(\nu^{F_\nu}_X(\mathrm{id}_X, f)\right) &= H^{\mathcal{P}'}_X(F_\nu(\mathrm{id}_X, f)) \\ &= H^{\mathcal{P}'}_X\left(\mathrm{id}_X, \nu_{\mathrm{id}_X}(y)^{-1} \circ \nu_X(f_2)\right) \\ &= i_X(\nu_X(y)) \circ \nu_{\mathrm{id}_X}(y)^{-1} \circ \nu_X(f) \end{aligned}$$

  We need to check that the following diagram commutes



  note that the lower square commutes by a coherence condition on pasting diagrams, and the upper triangle trivially commutes.

∎

Lemma J.3.6

$$\Phi \circ \Psi \cong \mathrm{id}_{\mathbf{Pfct}_{\mathcal{B}}}$$

Proof. We have exhibited a natural isomorphism

$$H : \Phi \circ \Psi \Longrightarrow \mathrm{id}_{\mathbf{Pfct}_{\mathcal{B}}}$$

□

J.3.2 $\Psi \circ \Phi$

Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration.

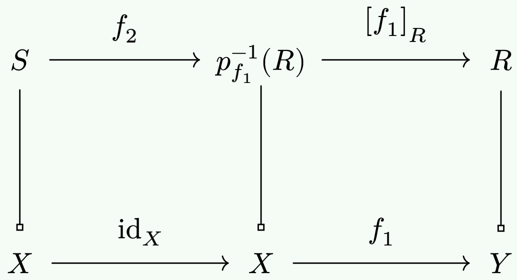
$$\Psi \circ \Phi(p) = \pi(p^{-1}) : \int p^{-1} \to \mathcal{B}$$

Definition J.3.7

Consider

$$\begin{aligned} G_p : \int p^{-1} &\longrightarrow \mathcal{E} \\ (X, R) &\longmapsto R \\ (f_1, f_2) &\longmapsto [f_1]_R \circ f_2 \end{aligned}$$

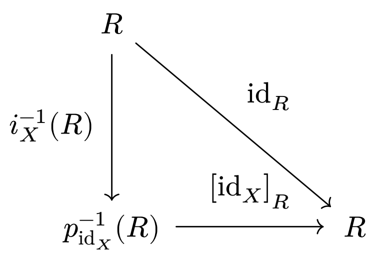
Let $(f_1, f_2) : (X, S) \to (Y, R)$, we have $f_1 : X \to Y$ and $f_2 : S \to p^{-1}_{f_1}(R)$



Lemma J.3.8

G_p is a functor.

Proof. Let $(X, R) : \int p^{-1}$. $\mathrm{id}_{X,R} = (\mathrm{id}_X, i_X^{-1}(R))$. We have to show that the following diagram commutes



which commutes by definition of i_X.

Furthermore, let $(X, R), (Y, S), (Z, T) : \int p^{-1}$, and

$$(f_1, f_2) : (X, R) \longrightarrow (Y, S)$$
$$(g_1, g_2) : (Y, S) \longrightarrow (Z, T)$$

Let us show that $G_p((g_1, g_2) \circ (f_1, f_2)) = G_p(g_1, g_2) \circ G_p(f_1, f_2)$, that is, that the following diagram commutes

$$\begin{array}{ccc}
R & \xrightarrow{\quad f_2 \quad} & p_{f_1}^{-1}(S) \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle [f_1]_S} \\
p_{f_1}^{-1}(S) & & S \\
{\scriptstyle p_{f_1}^{-1}(g_2)}\downarrow & & \downarrow{\scriptstyle g_2} \\
p_{f_1}^{-1}\left(p_{g_1}^{-1}(T)\right) & & p_{g_1}^{-1}(T) \\
{\scriptstyle [f_1, g_1]_R^{-1}}\downarrow & & \downarrow{\scriptstyle [g_1]_T} \\
p_{g_1 \circ f_1}^{-1}(T) & \xrightarrow{\quad [g \circ f]_T \quad} & T
\end{array}$$

We indeed have the following diagram commutes

$$\begin{array}{ccc}
R & \xrightarrow{\quad f_2 \quad} & p_{f_1}^{-1}(S) \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle [f_1]_S} \\
p_{f_1}^{-1}(S) & \xrightarrow{\quad [f_1]_S \quad} & S \\
{\scriptstyle p_{f_1}^{-1}(g_2)}\downarrow & & \downarrow{\scriptstyle g_2} \\
p_{f_1}^{-1}\left(p_{g_1}^{-1}(T)\right) & \xrightarrow{\quad [f_1]_{p_{g_1}^{-1}(T)} \quad} & p_{g_1}^{-1}(T) \\
{\scriptstyle [f_1, g_1]_R^{-1}}\downarrow & & \downarrow{\scriptstyle [g_1]_T} \\
p_{g_1 \circ f_1}^{-1}(T) & \xrightarrow{\quad [g \circ f]_T \quad} & T
\end{array}$$

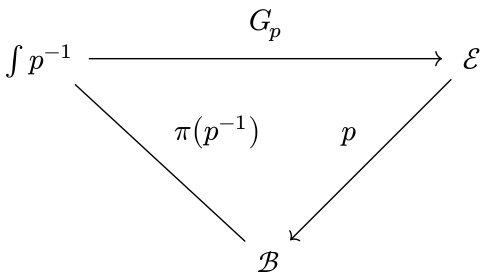
as the lower square commutes by definition of $[f_1, g_1]_R$, the middle one by definition of $p_{f_1}^{-1}(g_2)$, and the top one commutes trivially. □

Lemma J.3.9

> G_p is a fibration morphism.

Proof. There are two things to check: the commutation with the fibrations, and the preservation of cartesian morphisms. Let's proceed in order.

1.



Let's check that the two functors agree on objects and morphisms.

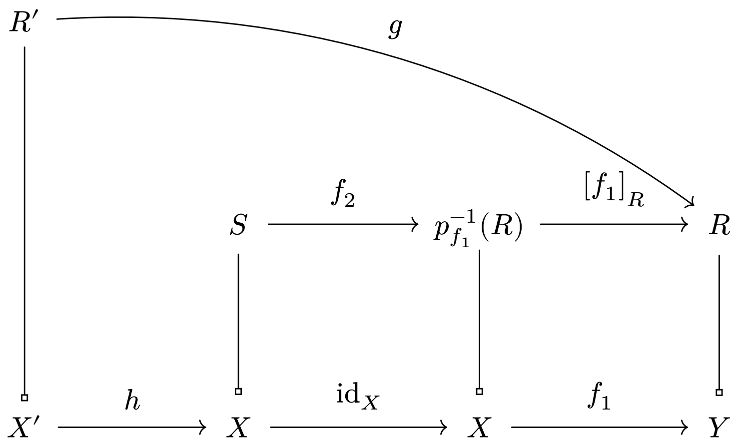
- let $(X, x) : \int p^{-1}$, ie $X = p(x)$

$$\begin{aligned} p\big(G_p(X, x)\big) &= p(x) \\ &= X \\ &= \pi(p^{-1})(X, x) \end{aligned}$$

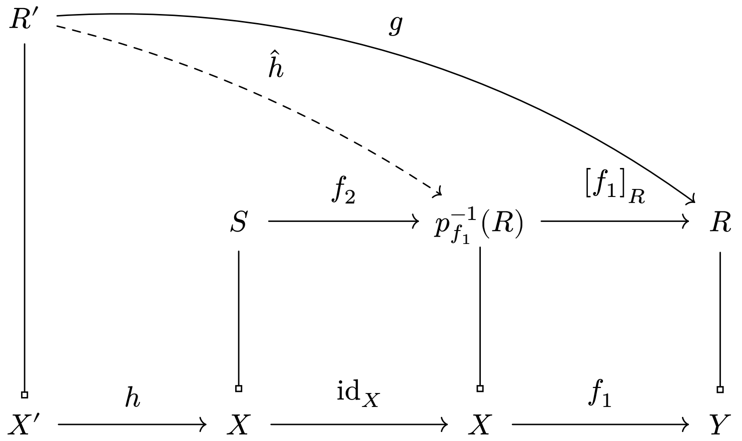
- let $(X, x), (Y, y) : \int p^{-1}$, and $(f_1, f_2) : (X, x) \to (Y, y)$. We have

$$\begin{aligned} p\big(G_p(f_1, f_2)\big) &= p\Big([f_1]_y \circ f_2\Big) \\ &= p\Big([f_1]_y\Big) \circ p(f_2) \\ &= f_1 \circ \mathrm{id}_X \\ &= f_1 \\ &= \pi(p^1)(f_1, f_2) \end{aligned}$$

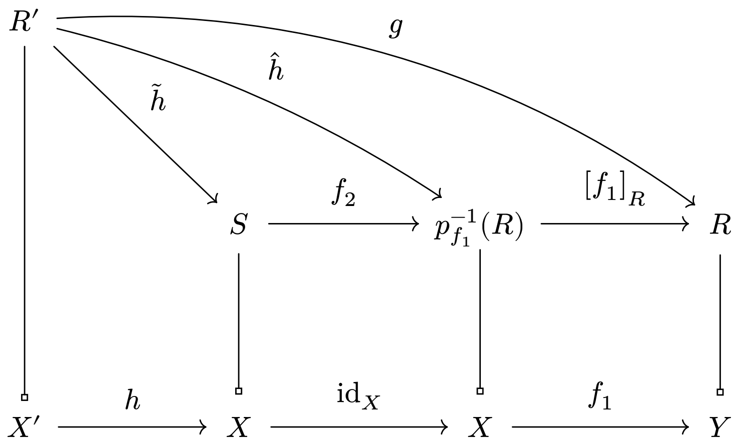
2. Let $(f_1, f_2) : (X, S) \to (Y, R)$ be a cartesian morphism. By Lemma J.2.5, f_2 is an isomorphism. Let $h : X' \to X$ and $g : R' \to R$ such that the following diagram commutes



By cartesianity of $[f_1]_R$, there exists a unique $\hat{h} : R' \to p^{-1}_{f_1}(R)$ such that the following diagram commutes



Hence, $f_2^{-1} \circ \hat{h}$ satisfies the wanted property. Furthermore, for any $\tilde{h} : R' \to S$ that makes the following diagram commute



note that $f_2 \circ \tilde{h}$ satisfies the same universal property as $\hat{h}$, hence $f_2 \circ \tilde{h} = \hat{h}$, and thus

$$\tilde{h} = f_2^{-1} \circ \hat{h}$$

which shows the unicity.

□

Lemma J.3.10

G_p is an isomorphism.

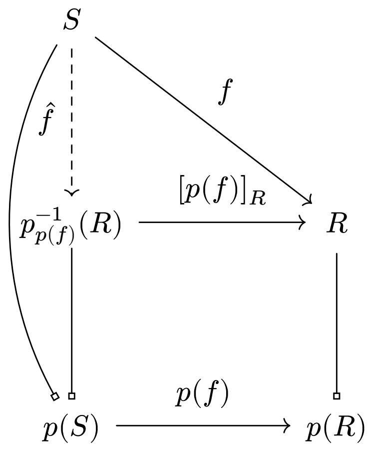
Proof. Let us exhibit an inverse morphism

$$K_p : \mathcal{E} \longrightarrow \int p^{-1}$$

- if $R : \mathcal{E}$, we define

$$K_p(R) = (p(R), R)$$

- if $S, R : \mathcal{E}$ and $f : S \longrightarrow R$ is a morphism in $\mathcal{E}$, by cartesianity of $[p(f)]_R$, there exists a unique $\hat{f} : S \to p_{p(f)}^{-1}(R)$ such that the following diagram commutes

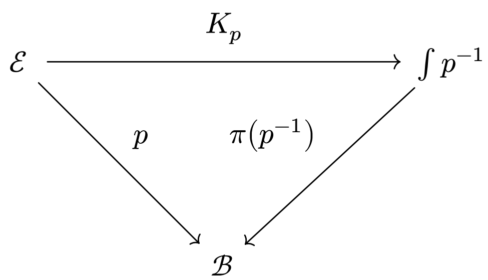


Let

$$K_p = \big(p(f), \hat{f}\big)$$

Let us show that K_p is the inverse of G_p (which will entail that it is a functor), and that it is a fibration morphism.

1. It is clear that K_p and G_p are each other's inverse on objects. Let (f_1, f_2) be a morphism in $\int p^{-1}$. We have that $p\big([f_1]_R \circ f_2\big) = p\big([f_1]_R\big) \circ p(f_2) = f_1 \circ \mathrm{id} = f_1$. Furthermore, f_2 is precisely the cartesian lifting of the identity by $[f_1]_R$, so we have $K_p\big(G_p(f_1, f_2)\big) = (f_1, f_2)$. Conversely, let $f : S \to R$ be a morphism in $\mathcal{E}$. By definition of $\hat{f}$, we have $f = [p(f)] \circ \hat{f}$, so $G_p\big(K_p(f)\big) = f$.
2. We have to check that the following diagram commutes



Let's check that the two functors agree on objects and morphisms.

- Let $R : \mathcal{E}$

$$\begin{aligned} \pi(p^1)\big(K_p(R)\big) &= \pi(p^{-1})(p(R), R) \\ &= p(R) \end{aligned}$$

- Let $S, R : \mathcal{E}$ and $f : S \to R$ a morphism in $\mathcal{E}$

$$\begin{aligned} \pi(p^{-1})\big(K_p(f)\big) &= \pi(p^{-1})\big(p(f), \hat{f}\big) \\ &= p(f) \end{aligned}$$

Furthermore, we have to check that K_p preserves cartesian morphisms. Let f be cartesian. $\hat{f}$ is (the canonical) isomorphism between the domains living in the same fiber, of two cartesian morphisms. In particular, it is an isomorphism, hence $\big(p(f), \hat{f}\big)$ is cartesian by Lemma J.2.6.

□

Lemma J.3.11

G_p is natural in p.

Proof. Let $p : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{F} \to \mathcal{B}$ be two fibrations, and $F : p \to q$ be a morphism of fibrations. Let's check that the following diagram commutes

$$\begin{array}{ccc} \int p^{-1} & \xrightarrow{G_p} & \mathcal{E} \\ {\scriptstyle F_{\nu^F}} \downarrow & & \downarrow {\scriptstyle F} \\ \int q^{-1} & \xrightarrow{G_q} & \mathcal{F} \end{array}$$

Let's check that the two functors agree on objects and morphisms. Let $(X, R) : \int p^{-1}$.

$$\begin{aligned} G_q(F_{\nu^F}(X, R)) &= G_q(X, \nu^F(R)) \\ &= \nu^F_X(R) \\ &= F(R) \\ &= F\big(G_p(X, R)\big) \end{aligned}$$

Let $(X, R), (Y, S) : \int p^{-1}$ and $(f_1, f_2) : (X, R) \to (Y, S)$ be a morphism in $\int p^{-1}$.

$$\begin{aligned} G_q(F_{\nu^F}(f_1, f_2)) &= G_q\Big(f_1, \nu_{f_1^F}(S)^{-1} \circ \nu^F_X(f_2)\Big) \\ &= [f_1]_{F(R)} \circ \nu_{f_1^F}(S)^{-1} \circ \nu^F_X(f_2) \\ &= [f_1]_{F(R)} \circ \nu_{f_1^F}(S)^{-1} \circ F(f_2) \\ &= F\Big([f_1]_R\Big) \circ F(f_2) \\ &= F\Big([f_1]_R \circ f_2\Big) \\ &= F\Big(G_p(f_1, f_2)\Big) \end{aligned}$$

□

Lemma J.3.12

$$\Psi \circ \Phi \cong \mathrm{id}_{\mathbf{Fib}_{\mathcal{B}}}$$

Proof. We have exhibited the natural isomorphism

$$G : \Psi \circ \Phi \Longrightarrow \mathrm{id}_{\mathbf{Fib}_{\mathcal{B}}}$$

□ This concludes the proof of Theorem A.3.1.