Polynomial functors

Models of type theories

Adrien Mathieu

2025-09-05

Hanabi

Fibrational Models of Linear Dependent Type Theories

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Dependent type theory

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Type theories, and in particular dependent type theories, are much more amenable to proof mechanization.

Dependent type theory

Homotopy Type Theory

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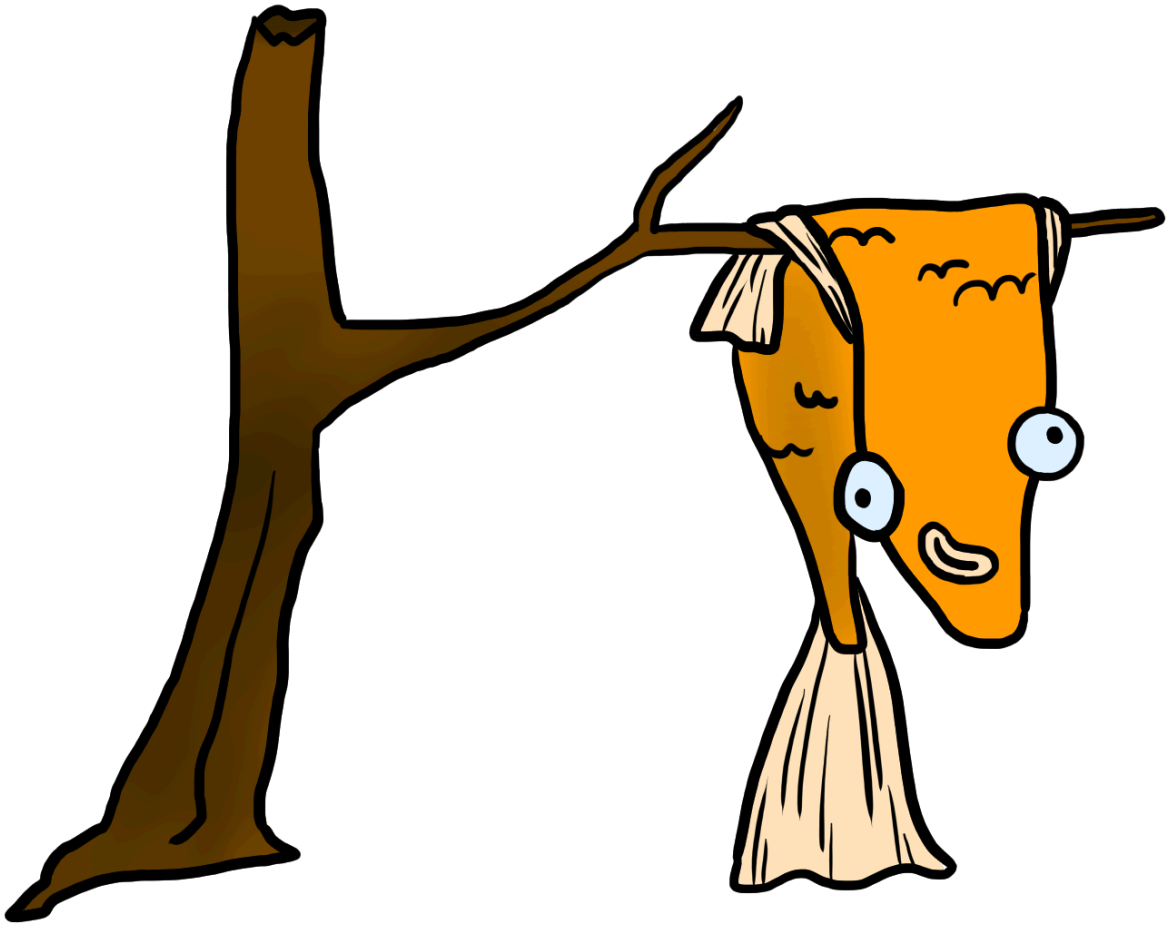
Solution:



Linear logic

Double negation elimination

Unfortunately, type theories build up *intuitionistic* mathematics, in which it is not necessarily true that, for any proposition P, we have

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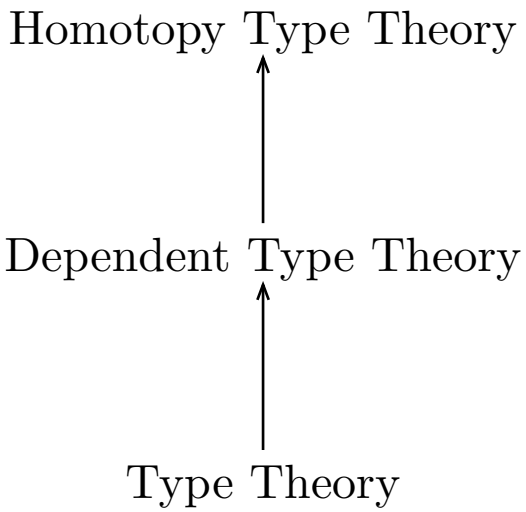

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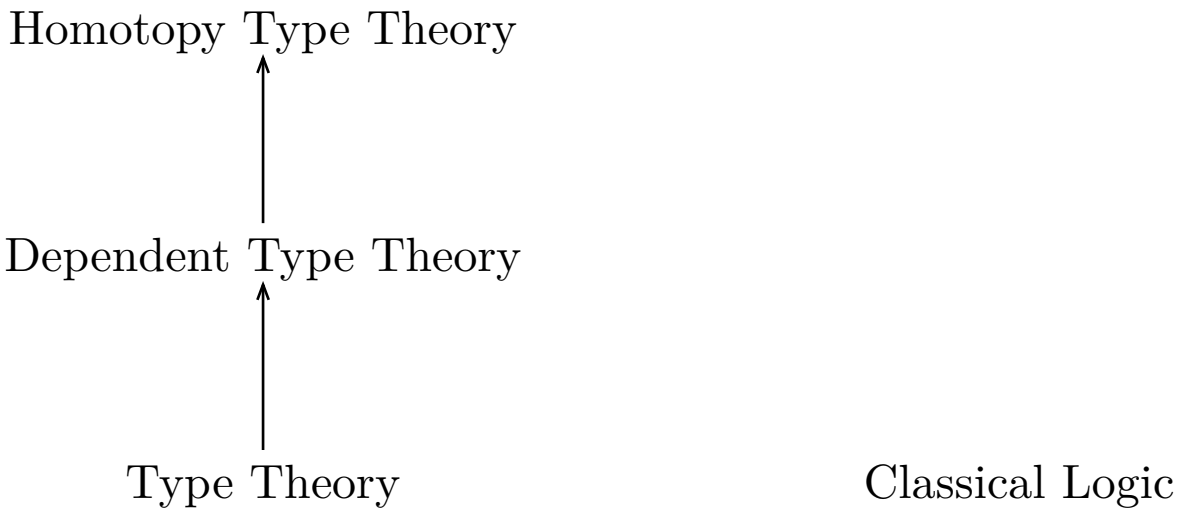

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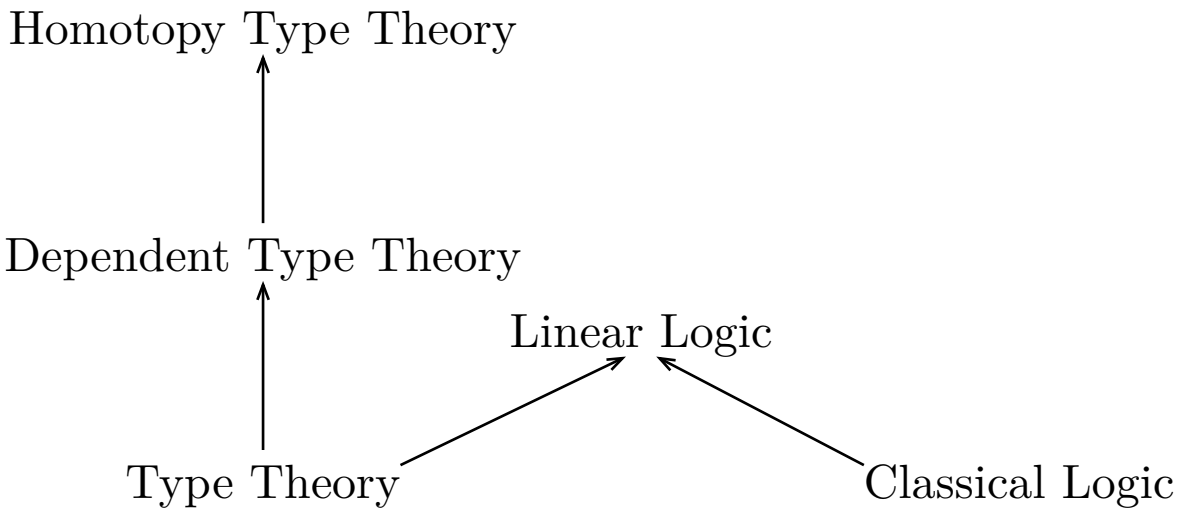


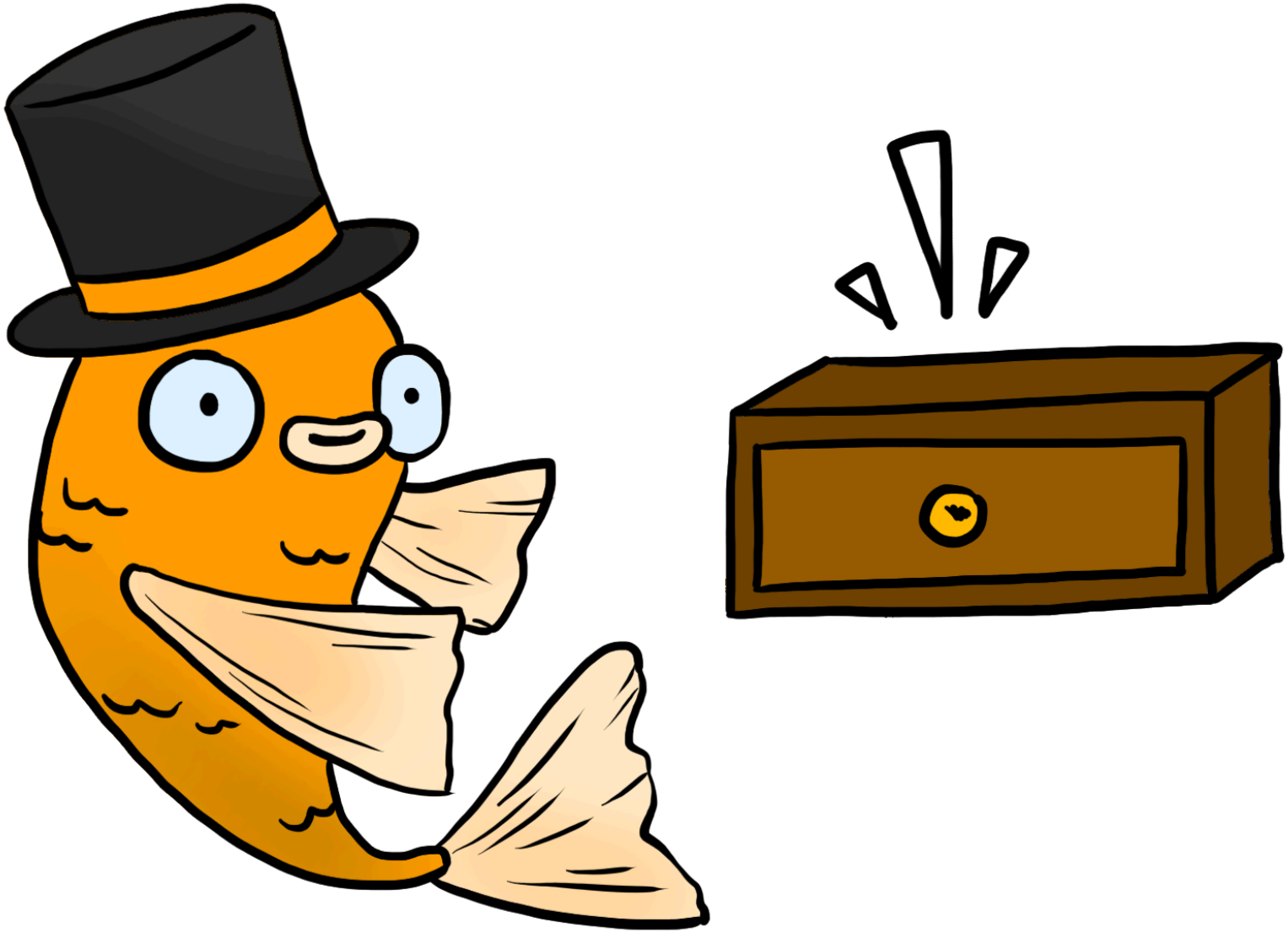
Linear logic

Linear Dependent Type Theory

What we need is a Dependent Linear Type Theory. There are, *a priori*, lots of possibilities to unify DTT and LL. However,

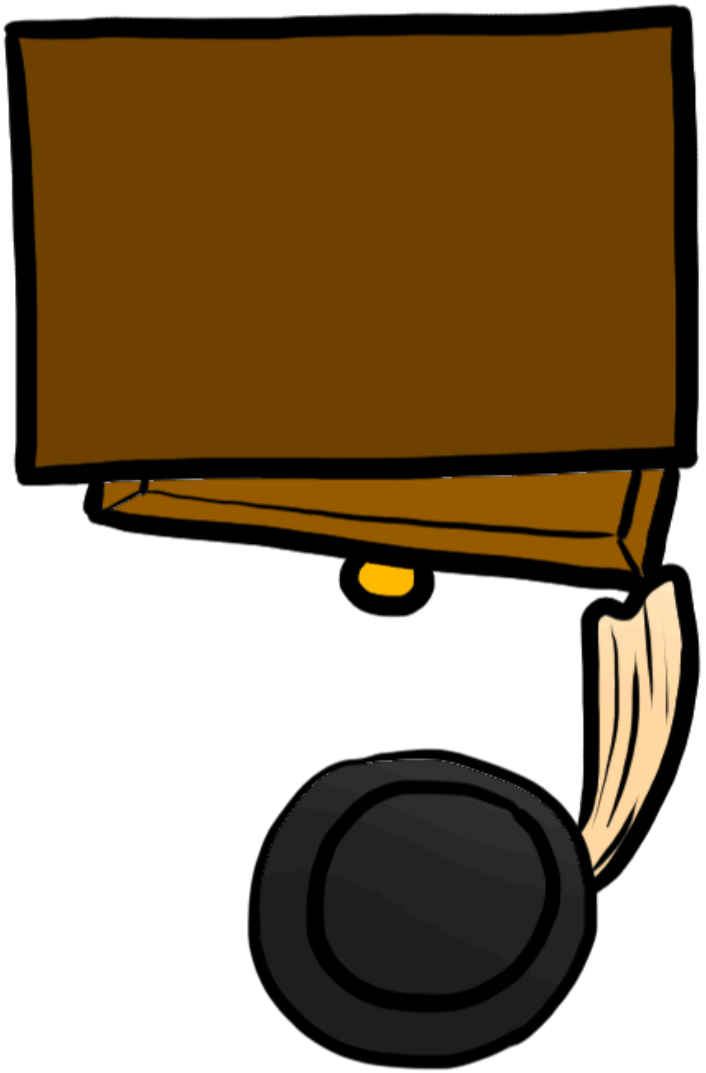
Work during the internship

We have a problem to solve.



Work during the internship

We try the syntaxic part. It doesn't fully work.



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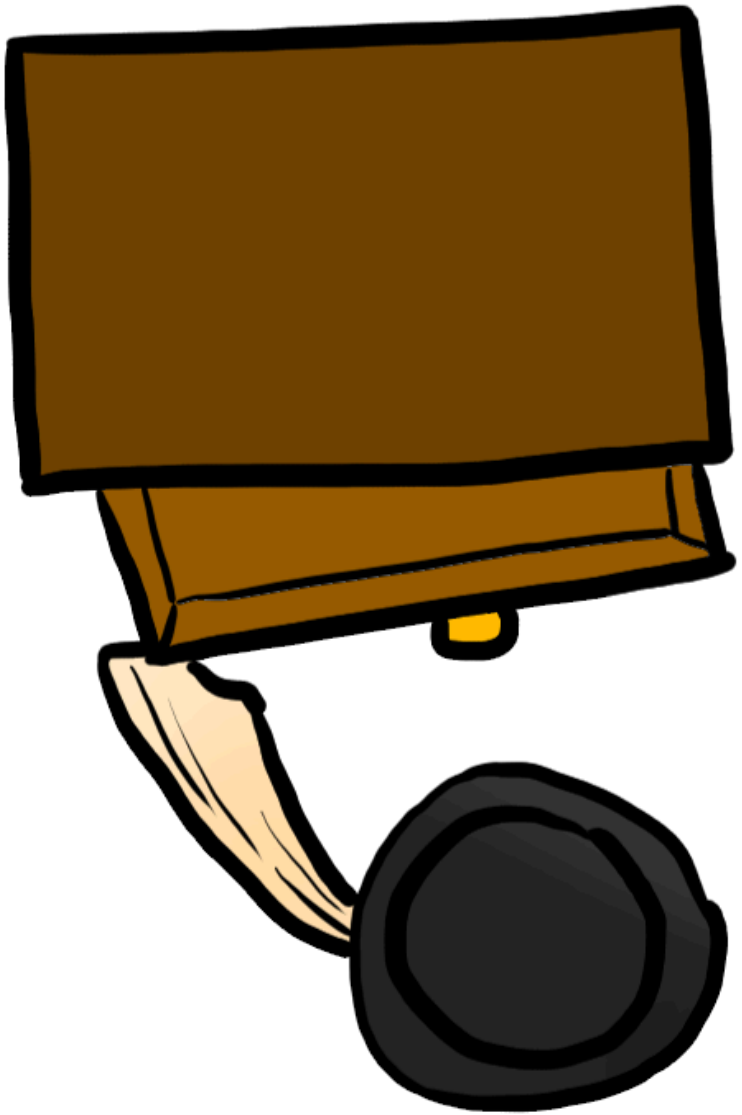
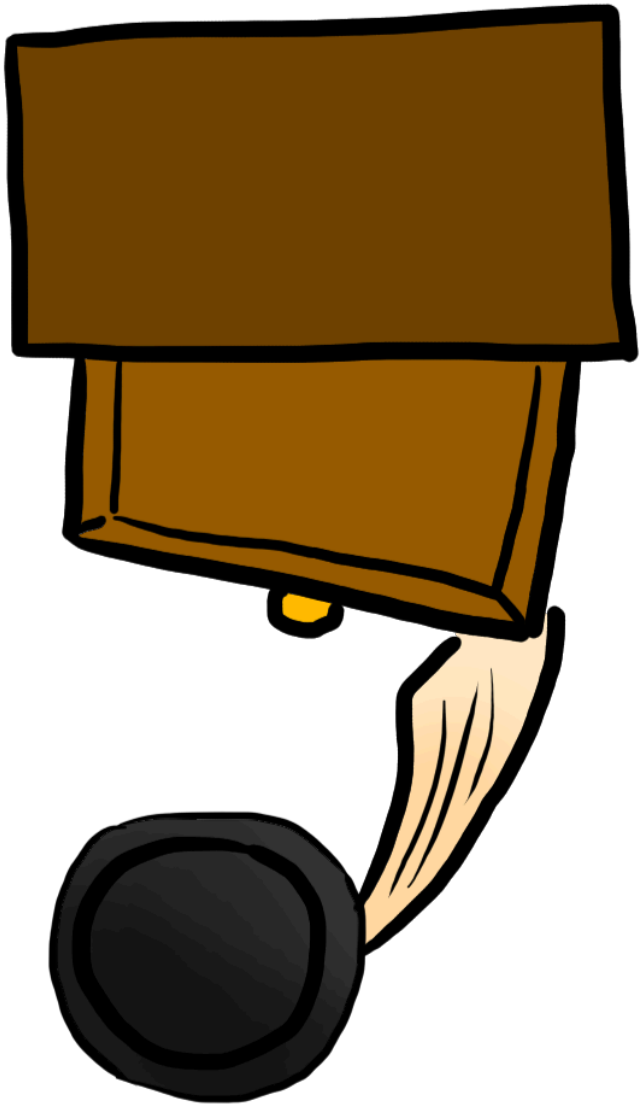
So we pull from the semantic part. We make some progress.



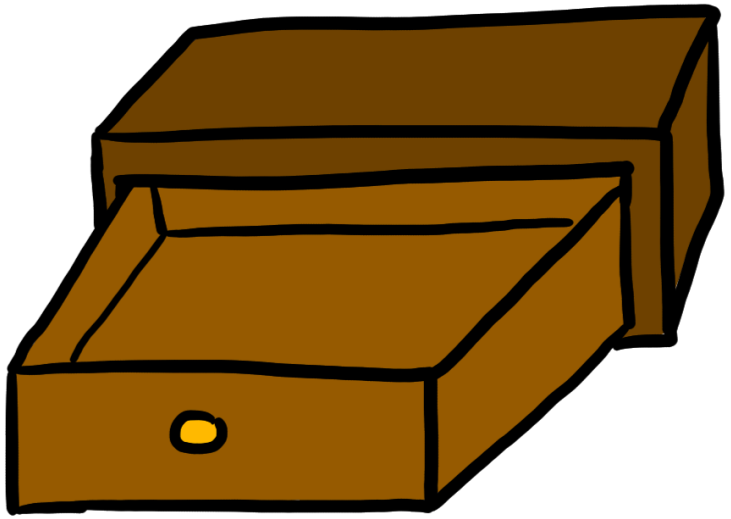
Work during the internship

We go back to the syntactic point of view, and keep pulling.



Work during the internship

The problem is solved.



Category theory reminders

Categories and functors

Definition (Category) A category $\mathcal{C}$ is

- a collection of objects $\mathcal{C}$
- for $X, Y : \mathcal{C}$, a collection of morphisms $\mathcal{C}(X, Y)$
- for any $X : \mathcal{C}$, an *identity at* X: $\mathrm{id}_X : \mathcal{C}(X, X)$
- a composition operation $\circ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$
- that is associative: $(f \circ g) \circ h = f \circ (g \circ h)$
- such that $\mathrm{id} \circ f = f \circ \mathrm{id} = f$

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Pullbacks

A square as follows

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

Category theory reminders

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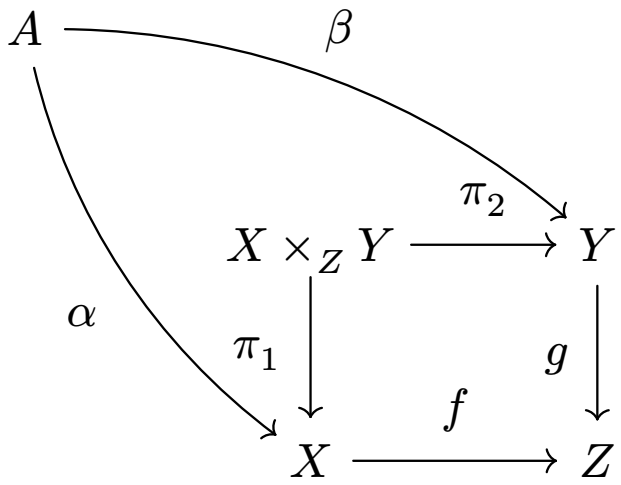

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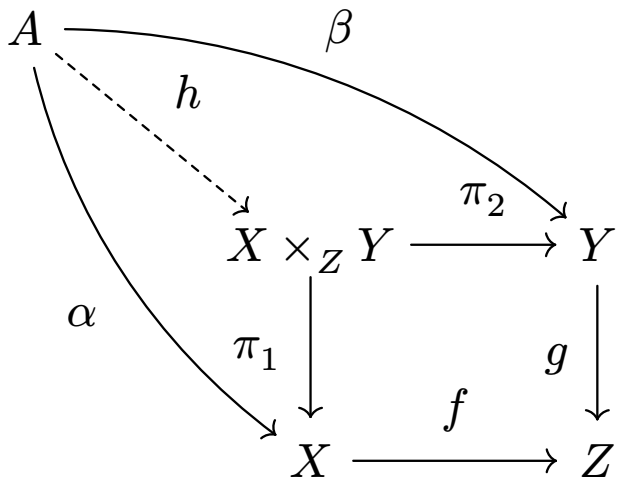

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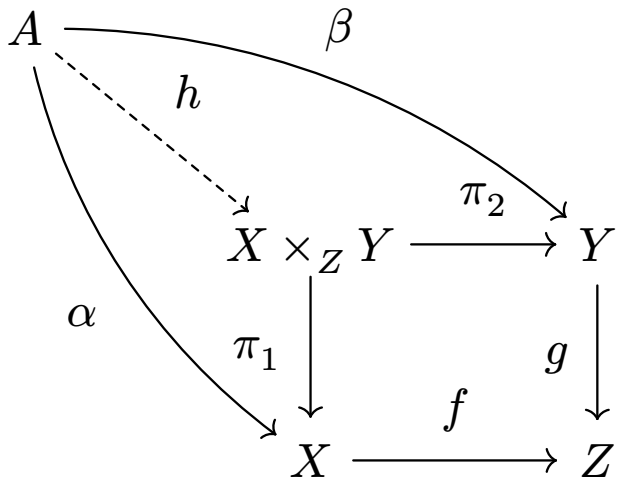


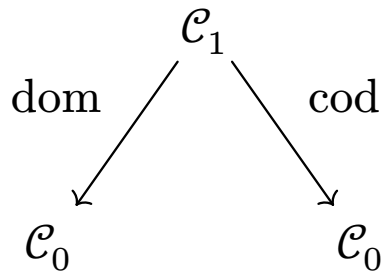
Example. If we are given two functions $f : X \to Z$ and $g : Y \to Z$, their pullback in **Set** is

$$\begin{array}{ccc} \{(x,y); x \in X, y \in Y \mid f(x) = g(y)\} & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

Double categories

Category theory, again

Let's repeat what a category is. A category $\mathcal{C}$ is a collection of objects $\mathcal{C}_0$ and a collection of morphisms $\mathcal{C}_1$, with two functions



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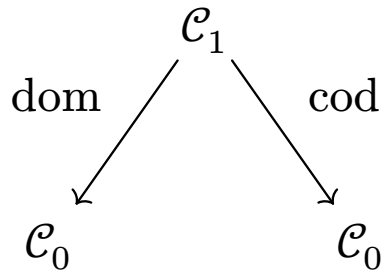

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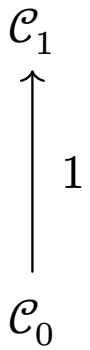

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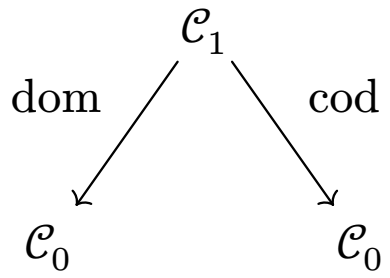

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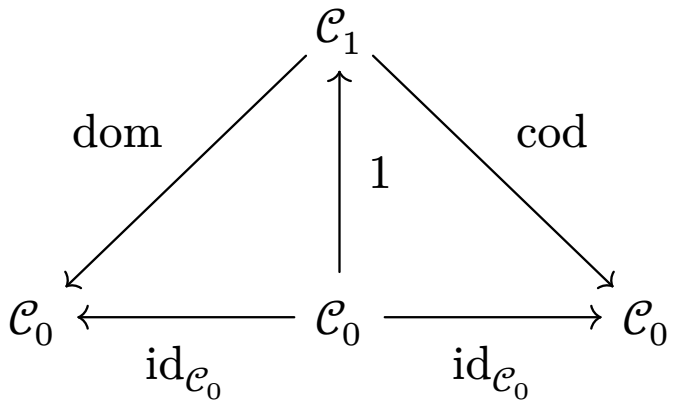

Double categories

Category theory, again

we also need a composition

$$\begin{array}{c} \mathcal{C}_1 \\ \uparrow \\ \{(f,g) \mid \mathrm{cod}(f) = \mathrm{dom}(g)\} \end{array} \circ$$

Double categories

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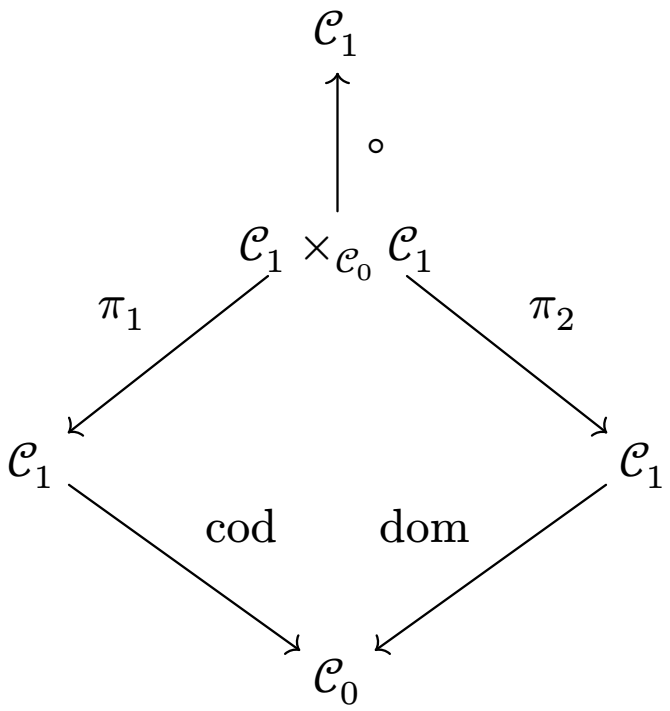

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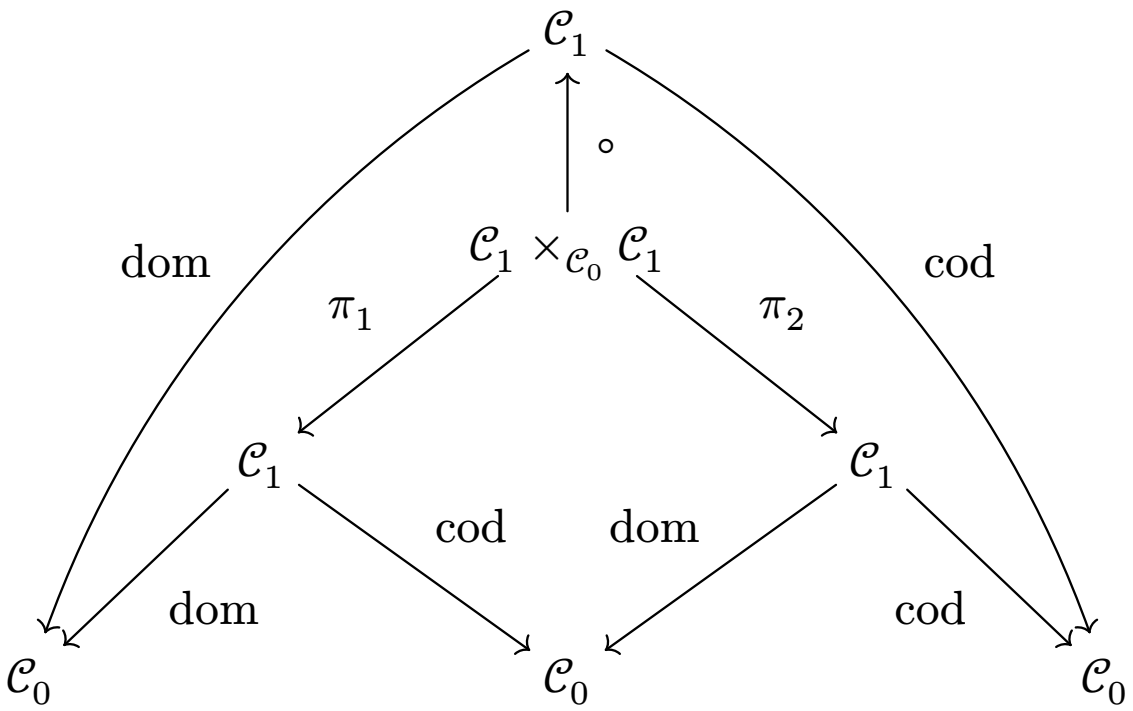

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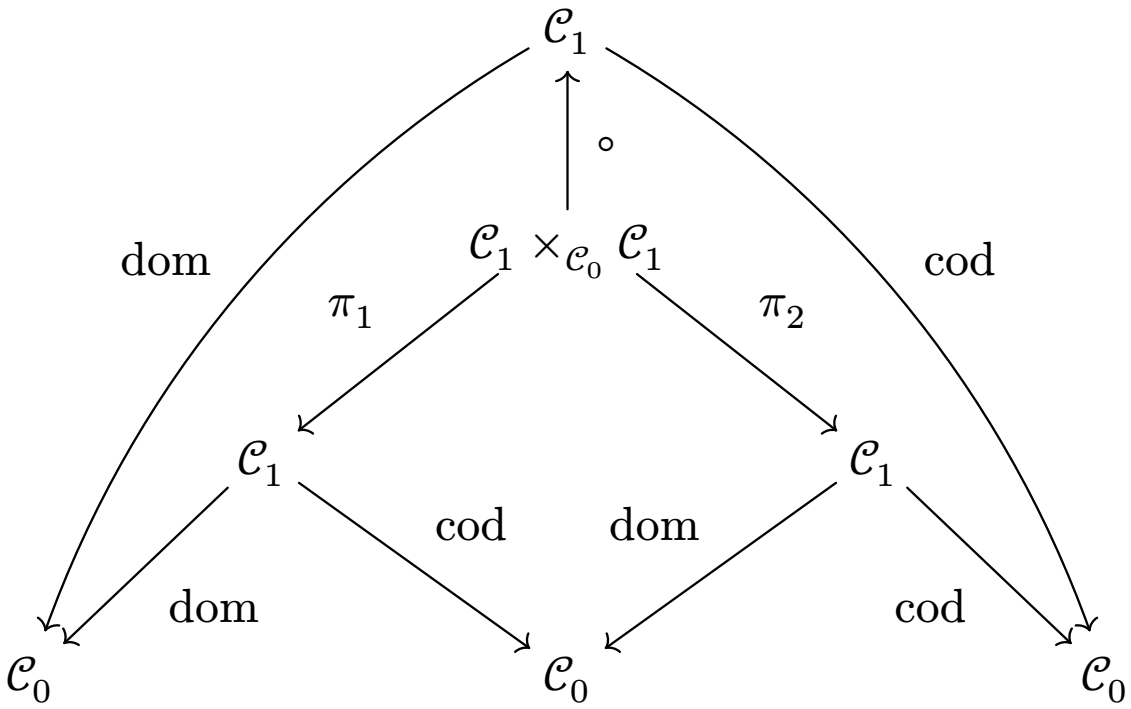
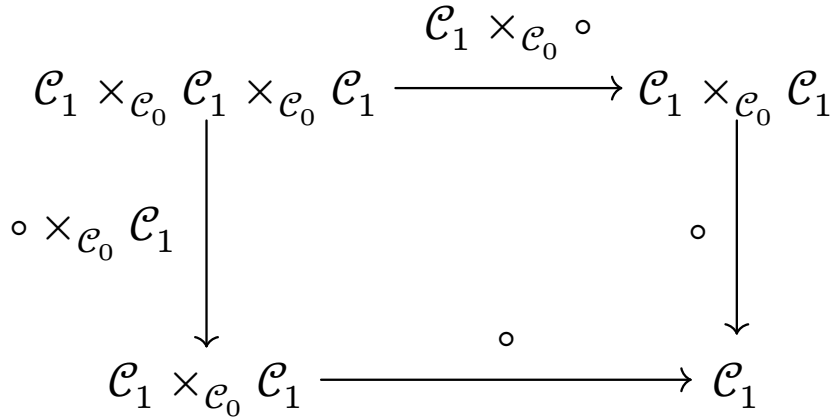


that is associative



Double categories

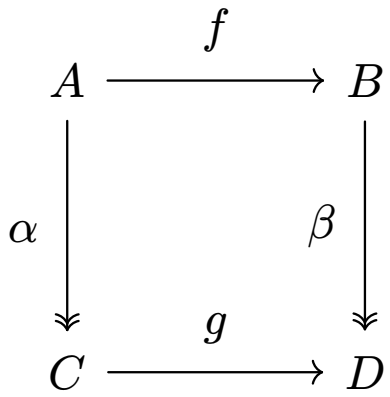
Categories in categories

So… a category is a diagram in **Set**, making additional diagrams commute. But these diagrams make sense in any category with pullbacks. For instance, we can consider internal categories in **Cat**

Definition (Double category) A *double category* is an internal category in **Cat**.

> ⚠ **Warning**
>
> A double category is not a 2-category, or a bicategory!

A double category is like a category, but with two kinds of morphisms: vertical and horizontal morphisms. Each class of morphism behaves like a category, and additionally, a double category tells when the following diagrams "commute"



Double categories

Mental break

Fibrations

Refinement system

Suppose we have a functor $p : \mathcal{E} \to \mathcal{B}$. For $R : \mathcal{E}$ and $X : \mathcal{B}$, we note

$$\begin{array}{c} R \\ p \,\big|\, \\ X \end{array}$$

if $X = p(R)$.

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if $X = p(R)$.

Similarly, we say that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & S \\ p \,\big|\, & & \big|\, p \\ X & \xrightarrow{f} & Y \end{array}$$

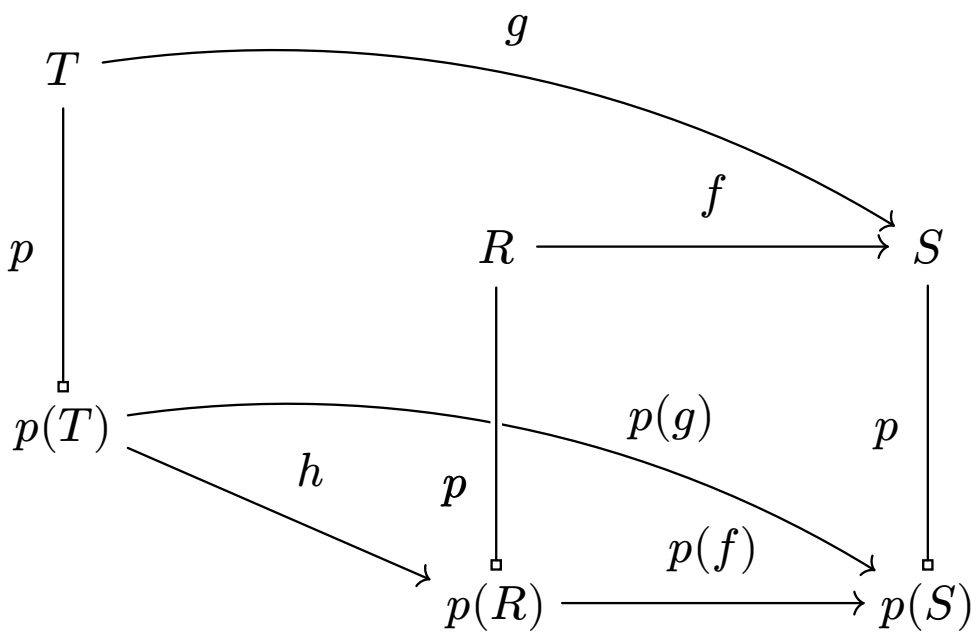
when

$$p(\alpha) = f$$

Fibrations

Cartesian morphism

A morphism $f : R \to S$ is *cartesian* if



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